PERIODIC ORBITS OF TWISTED GEODESIC FLOWS AND THE WEINSTEIN–MOSER THEOREM

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Abstract. In this paper, we establish the existence of periodic orbits of a twisted geodesic flow on all low energy levels and in all dimensions whenever the magnetic field form is symplectic and spherically rational. This is a consequence of a more general theorem concerning periodic orbits of autonomous Hamiltonian flows near Morse–Bott non-degenerate, symplectic extrema. Namely, we show that all energy levels near such extrema carry periodic orbits, provided that the ambient manifold meets certain topological requirements. This result is a partial generalization of the Weinstein–Moser theorem. The proof of the generalized Weinstein–Moser theorem is a combination of a Sturm–theoretic argument and a Floer homology calculation.

1. Introduction and main results

In the early 1980s, V.I. Arnold proved, as a consequence of the Conley–Zehnder theorem, [CZ1], the existence of periodic orbits of a twisted geodesic flow on $\mathbb{T}^2$ with symplectic magnetic field for all energy levels when the metric is flat and low energy levels for an arbitrary metric, [Ar2]. This result initiated an extensive study of the existence problem for periodic orbits of general twisted geodesic flows via Hamiltonian dynamical systems methods and in the context of symplectic topology, mainly focusing on low energy levels. (A brief and admittedly incomplete survey of some related work is provided in Section 1.3.)

In the present paper, we establish the existence of periodic orbits of a twisted geodesic flow on all low energy levels and in all dimensions whenever the magnetic field form is symplectic and spherically rational. An essential point is that, in contrast with other results of this type, we do not require any compatibility conditions on the Hamiltonian and the magnetic field. In fact, we prove a more general theorem concerning periodic orbits of autonomous Hamiltonian flows near Morse–Bott non-degenerate, symplectic extrema. Namely, we show that all energy levels near such extrema carry periodic orbits, provided that the ambient manifold meets certain topological requirements. This result is a (partial) generalization of the Weinstein–Moser theorem, [Mo, We1], asserting that a certain number of distinct periodic orbits exist on every energy level near a non-degenerate extremum. The proof of the generalized Weinstein–Moser theorem is a combination of a Sturm–theoretic argument utilizing convexity of the Hamiltonian in the direction normal to the critical submanifold and of a Floer–homological calculation that guarantees

Date: February 1, 2008.

2000 *Mathematics Subject Classification.* 53D40, 37J10, 37J45.

The work is partially supported by the NSF and by the faculty research funds of the University of California, Santa Cruz.

"dense existence" of periodic orbits with certain index. The existence of periodic orbits for a twisted geodesic flow with symplectic magnetic field is then an immediate consequence of the generalized Weinstein–Moser theorem.

1.1. **The generalized Weinstein–Moser theorem.** Throughout the paper, M will stand for a closed symplectic submanifold of a symplectic manifold (P, ω). We denote by $[\omega]$ the cohomology class of ω and by $c_1(TP)$ the first Chern class of P equipped with an almost complex structure compatible with ω. The integrals of these classes over a 2-cycle u will be denoted by $\langle \omega, u\rangle$ and, respectively, $\langle c_1(TP), u\rangle$. Recall also that P is said to be *spherically rational* if the integrals $\langle \omega, u\rangle$ over all $u \in \pi_2(P)$ are commensurate, i.e., $\lambda_0 = \inf\{|\langle \omega, u\rangle| \mid u \in \pi_2(P)\} > 0$ or, equivalently, $\langle \omega, \pi_2(P)\rangle$ is a discrete subgroup of $\mathbb{R}$.

The key result of the paper is

Theorem 1.1 (Generalized Weinstein–Moser theorem)**.** *Let $K\colon P \to \mathbb{R}$ be a smooth function on a symplectic manifold (P, ω), which attains its minimum $K = 0$ along a closed symplectic submanifold $M \subset P$. Assume in addition that the critical set M is Morse–Bott non-degenerate and one of the following cohomological conditions is satisfied:*

(i) *M is spherically rational and $c_1(TP) = 0$, or*
(ii) *$c_1(TP) = \lambda[\omega]$ for some $\lambda \neq 0$.*

Then for every sufficiently small $r^2 > 0$ the level $K = r^2$ carries a contractible in P periodic orbit of the Hamiltonian flow of K with period bounded from above by a constant independent of r.

When M is a point, Theorem 1.1 turns into the Weinstein–Moser theorem (see [We1] and [Mo]) on the existence of periodic orbits near a non-degenerate extremum, albeit without the lower bound $1 + \dim P/2$ on the number of periodic orbits.

Remark 1.2. The assertion of the theorem is local and concerns only a neighborhood of M in P. Hence, in (i) and (ii), we can replace $c_1(TP)$ by $c_1(TP|_M) = c_1(TM) + c_1(TM^\perp)$ and $[\omega]$ by $[\omega|_M]$. Also note that in (ii) we do not require λ to be positive, i.e., M need not be monotone. (However, this condition does imply that M is spherically rational.) We also emphasize that we do need conditions (i) and (ii) in their entirety – the weaker requirements $c_1(TP)|_{\pi_2(P)} = 0$ or $c_1(TP)|_{\pi_2(P)} = \lambda[\omega]|_{\pi_2(P)}$, common in symplectic topology, are not sufficient for the proof.

Although conditions (i) and (ii) enter our argument in an essential way, their role is probably technical (see Section 7.2), and one may expect the assertion of the theorem to hold without any cohomological restrictions on P. For instance, this is the case whenever $\operatorname{codim} M = 2$; see [Gi2]. Furthermore, when $\operatorname{codim} M \geq 2$ the theorem holds without (i) and (ii), provided that the normal direction Hessian $d^2_M K$ and ω meet a certain geometrical compatibility requirement; [GK1, GK2, Ke1]. On the other hand, the condition that the extremum M is Morse–Bott non-degenerate is essential; see [GG2].

1.2. **Periodic orbits of twisted geodesic flows.** Let M be a closed Riemannian manifold and let σ be a closed 2-form on M. Equip T^*M with the twisted symplectic structure $\omega = \omega_0 + \pi^*\sigma$, where ω_0 is the standard symplectic form on T^*M and $\pi\colon T^*M \to M$ is the natural projection. Denote by K the standard kinetic energy Hamiltonian on T^*M corresponding to a Riemannian metric on M. The

Hamiltonian flow of K on T^*M describes the motion of a charge on M in the *magnetic field* σ and is sometimes referred to as a magnetic or *twisted geodesic flow*; see, e.g., [Gi3] and references therein for more details. Clearly, $c_1(T(T^*M)) = 0$, for T^*M admits a Lagrangian distribution (e.g., formed by spaces tangent to the fibers of π), and M is a Morse–Bott non-degenerate minimum of K. Furthermore, M is a symplectic submanifold of T^*M when the form σ symplectic. Hence, as an immediate application of case (i) of Theorem 1.1, we obtain

Theorem 1.3. *Assume that σ is symplectic and spherically rational. Then for every sufficiently small $r^2 > 0$ the level $K = r^2$ carries a contractible in T^*M periodic orbit of the twisted geodesic flow with period bounded from above by a constant independent of r.*

Remark 1.4. The proof of Theorem 1.1 is particularly transparent when P is *geometrically bounded* and *symplectically aspherical* (i.e., $\omega|_{\pi_2(P)} = 0 = c_1(TP)|_{\pi_2(P)}$). This particular case is treated in Section 4, preceding the proof of the general case. The twisted cotangent bundle (T^*M, ω) is geometrically bounded; see [AL, CGK, Lu1]. Furthermore, (T^*M, ω) is symplectically aspherical if and only if (M, σ) is weakly exact (i.e., $\sigma|_{\pi_2(M)} = 0$).

Note also that, as the example of the horocycle flow shows, a twisted geodesic flow with symplectic magnetic field need not have periodic orbits on all energy levels; see, e.g., [CMP, Gi3] for a detailed discussion of this example and of the resulting transition in the dynamics from low to high energy levels. Similar examples also exist for twisted geodesic flows in dimensions greater than two, [Gi4, Section 4].

1.3. **Related results.** To the best of the authors' knowledge, the existence problem for periodic orbits of a charge in a magnetic field was first addressed by V.I. Arnold in the early 1980s; [Ar2, Ko]. Namely, V.I. Arnold established the existence of at least three periodic orbits of a twisted geodesic flow on $M = \mathbb{T}^2$ with symplectic magnetic field for all energy levels when the metric is flat and low energy levels for an arbitrary metric. (It is still unknown if the second of these results can be extended to all energy levels.) Since then the question has been extensively investigated. It was interpreted (for a symplectic magnetic field) as a particular case of the generalized Weinstein–Moser theorem in [Ke1]. Referring the reader to [Gi3, Gi6, Gi7] for a detailed review and further references, we mention here only some of the results most relevant to Theorems 1.1 and 1.3.

The problems of *almost existence* and *dense existence* of periodic orbits concern the existence of periodic orbits on almost all energy levels and, respectively, on a dense set of levels. In the setting of the generalized Weinstein–Moser theorem or of twisted geodesic flows, these problems are studied for low energy levels in, e.g., [CGK, Co, CIPP, FS, GG2, Gü, Ke3, Ma, Lu1, Lu2, Schl], following the original work of Hofer and Zehnder and of Struwe, [FHW, HZ1, HZ2, HZ3, St]. In particular, almost existence for periodic orbits near a symplectic extremum is established in [Lu2] under no restrictions on the ambient manifold P. When P is geometrically bounded and (stably) strongly semi-positive, almost existence is proved for almost all low energy levels in [Gü] under the assumption that $\omega|_M$ does not vanish at any point, and in [Schl] when M has middle-dimension and $\omega|_M \neq 0$. These results do not require the extremum M to be Morse–Bott non-degenerate. Very strong almost existence results (not restricted to low energy levels) for twisted geodesic flows with exact magnetic fields and also for more general Lagrangian systems

are obtained in [Co, CIPP]. The dense or almost existence results established in [CGK, GG2, Ke3] follow from Theorem 1.1. However, the proof of Theorem 1.1 relies on the almost existence theorem from [GG2] or, more precisely, on the underlying Floer homological calculation.

As is pointed out in Section 1.1, in the setting of the generalized Weinstein–Moser theorem without requirements (i) and (ii), every low energy level carries a periodic orbit whenever $\operatorname{codim} M = 2$ or provided that the normal direction Hessian $d^2_M K$ and ω meet certain geometrical compatibility conditions, which are automatically satisfied when $\operatorname{codim} M = 2$ or M is a point; see [Gi1, Gi2, GK1, GK2, Ke1, Mo, We1] and references therein. Moreover, under these conditions, non-trivial lower bounds on the number of distinct periodic orbits have also been obtained. The question of existence of periodic orbits of twisted geodesic flows on (low) energy levels for magnetic fields on surfaces is studied in, e.g., [No, NT, Ta1, Ta2] in the context of Morse–Novikov theory; see also [Co, CIPP, CMP, Gi6] for further references. (In general, this approach requires no non-degeneracy condition on the magnetic field.) For twisted geodesic flows on surfaces with exact magnetic fields, existence of periodic orbits on all energy levels is proved in [CMP].

1.4. **Infinitely many periodic orbits.** The multiplicity results from [Ar2, Gi1, Gi2, GK1, GK2, Ke1] rely (implicitly in some instances) on the count of "short" periodic orbits of the Hamiltonian flow on $K = r^2$. The resulting lower bounds on the number of periodic orbits can be viewed as a "crossing-over" between the Weinstein–Moser type lower bounds in the normal direction to M and the Arnold conjecture type lower bounds along M. This approach encounters serious technical difficulties unless ω and $d^2_M K$ meet some geometrical compatibility requirements, for otherwise even identifying the class of short orbits is problematic.

However, looking at the question from the perspective of the Conley conjecture (see [FrHa, Gi9, Hi, SZ]) rather than of the Arnold conjecture, one can expect every low level of K to carry infinitely many periodic orbits (not necessarily short), provided that $\dim M \geq 2$ and M is symplectically aspherical. An indication that this may indeed be the case is given by

Proposition 1.5. *Assume that M is symplectically aspherical and not a point, and* $\operatorname{codim} M = 2$ *and the normal bundle to M in P is trivial. Then every level $K = r^2$, where $r > 0$ is sufficiently small, carries infinitely many distinct, contractible in P periodic orbits of K.*

This proposition does not rely on Theorem 1.1 and is an immediate consequence of the results of [Ar2, Gi1] and the Conley conjecture; see [Gi9] and also [FH, Hi, SZ]. For the sake of completeness, a detailed argument is given in Section 4.4. In a similar vein, in the setting of Theorem 1.3 with $M = \mathbb{T}^2$ and K arising from a flat metric, the level $K = r^2$ carries infinitely many periodic orbits for every (not necessarily small) $r > 0$.

1.5. **Outline of the proof of Theorem 1.1 and the organization of the paper.** The proof of Theorem 1.1 hinges on an interplay of two counterparts: a version of the Sturm comparison theorem and a Floer homological calculation. Namely, on the one hand, a Floer homological calculation along the lines of [GG2] guarantees that almost all low energy levels of K carry periodic orbits with Conley–Zehnder index depending only on the dimensions of P and M. On the other hand, since the levels of K are fiber-wise convex in a tubular neighborhood of M, a

Sturm theoretic argument ensures that periodic orbits with large period must also have large index. (Strictly speaking, the orbits in question are degenerate and the Conley–Zehnder index is not defined. Hence, we work with the Salamon–Zehnder invariant Δ, [SZ], but the Robin–Salamon index, [RS], could be utilized as well.) Thus, the orbits detected by Floer homology have period *a priori* bounded from above and the existence of periodic orbits on all levels follows from the Arzela–Ascoli theorem.

The paper is organized as follows. In Section 2, we recall the definition and basic properties of the Salamon–Zehnder invariant Δ and also prove a version of the Sturm comparison theorem giving a lower bound for the growth of Δ in linear systems with positive definite Hamiltonians. This lower bound is extended to periodic orbits of K near M in Propositions 3.1 and 3.2 of Section 3, providing the Sturm–theoretic counterpart of the proof of Theorem 1.1. In Section 4, we prove Theorem 1.1 under the additional assumptions that P is geometrically bounded and symplectically aspherical. In this case, clearly illustrating the interplay between Sturm theory and Floer homology, we can directly make use of a Floer homological calculation from [GG2]. Turning to the general case, we define in Section 5 a version of filtered Floer (or rather Floer–Novikov) homology of compactly supported Hamiltonians on open manifolds. The relevant part of the calculation from [GG2] is extended to the general setting in Section 6. The proof of Theorem 1.1 is completed in Section 7 where we also discuss some other approaches to the problem. Proposition 1.5 is proved in Section 4.4.

Acknowledgements. The authors are deeply grateful to Michael Entov, Ely Kerman, and Leonid Polterovich for valuable discussions, remarks and suggestions.

2. The Salamon–Zehnder invariant Δ

In this section we briefly review the properties of the invariant Δ, a continuous version of the Conley–Zehnder index introduced in [SZ], used in the proof of Theorem 1.1.

2.1. Linear algebra.

Let (V, ω) be a symplectic vector space. Throughout this paper we denote the group of linear symplectic transformations of V by $\mathrm{Sp}(V, \omega)$ or simply $\mathrm{Sp}(V)$ when the form ω is clear from the context. Moreover, if V is also equipped with a complex structure J we will use the notation $\mathrm{U}(V, \omega, J)$ or just $\mathrm{U}(V)$ for the group of unitary transformations, i.e., transformations preserving J and ω. For $A \in \mathrm{U}(V)$ we denote by $\det_{\mathbb{C}} A \in S^1$ the complex determinant of A.

Salamon and Zehnder, [SZ], proved that there exists a unique collection of continuous maps $\rho\colon \mathrm{Sp}(V, \omega) \to S^1 \subset \mathbb{C}$, where (V, ω) ranges through all finite-dimensional symplectic vector spaces, with the following properties:

- For any $A \in \mathrm{Sp}(V, \omega)$ and any linear isomorphism $B\colon W \to V$, we have $\rho(B^{-1}AB) = \rho(A)$. (Note that $B^{-1}AB \in \mathrm{Sp}(W, B^*\omega)$.) In particular, ρ is conjugation invariant on $\mathrm{Sp}(V, \omega)$.
- Whenever $A_1 \in \mathrm{Sp}(V_1, \omega_1)$ and $A_2 \in \mathrm{Sp}(V_2, \omega_2)$, we have $\rho(A_1 \times A_2) = \rho(A_1)\rho(A_2)$, where $A_1 \times A_2$ is viewed as a symplectic transformation of $(V_1 \times V_2, \omega_1 \times \omega_2)$.
- For $A \in \mathrm{U}(V, \omega, J)$, we have $\rho(A) = \det_{\mathbb{C}} A$.
- For A without eigenvalues on the unit circle, $\rho(A) = \pm 1$.

Note that $\rho(A)$ is completely determined by the eigenvalues of A together with a certain "ordering" of eigenvalues, and in fact only the eigenvalues of A on the unit circle matter. It is also worth emphasizing that ρ is not smooth on $\mathrm{Sp}(V)$. Furthermore, although in general $\rho(AB) \neq \rho(A)\rho(B)$, we have

$$\rho(A^k) = \rho(A)^k$$

for all $k \in \mathbb{Z}$. In particular, $\rho(A^{-1}) = \overline{\rho(A)}$.

2.2. **The Salamon–Zehnder quasi-morphism Δ.**

2.2.1. *Definition and basic properties.* In this section we recall the definition and basic properties of the *Salamon–Zehnder invariant* Δ following closely [SZ]. Let $\Phi\colon [a, b] \to \mathrm{Sp}(V)$ be a continuous path. Pick a continuous function $\lambda\colon [a, b] \to \mathbb{R}$ such that $\rho(\Phi(t)) = e^{2\pi i \lambda(t)}$ and set

$$\Delta(\Phi) = \frac{\lambda(b) - \lambda(a)}{\pi} \in \mathbb{R}.$$

It is clear that $\Delta(\Phi)$ is independent of the choice of λ and that geometrically $\Delta(\Phi)$ measures the total angle swept by $\rho(\Phi(t))$ as t varies from a to b. Note also that we do not require $\Phi(a)$ to be the identity transformation.

As an immediate consequence of the definition, $\Delta(\Phi)$ is an invariant of homotopy of Φ with fixed end-points. In particular, Δ gives rise to a continuous map $\widetilde{\mathrm{Sp}}(V) \to \mathbb{R}$, where $\widetilde{\mathrm{Sp}}(V)$ is the universal covering of $\mathrm{Sp}(V)$. Furthermore, $\Delta(\Phi)$ is an invariant of (orientation preserving) reparametrizations of Φ. On the other hand, let Φ^{inv} be the path Φ traversed in the opposite direction. Then

$$\Delta(\Phi^{\mathrm{inv}}) = \Delta(\Phi^{-1}) = -\Delta(\Phi).$$

Finally, Δ is additive with respect to concatenation of paths. More explicitly, assume that $a < c < b$. Then, in obvious notation,

$$\Delta(\Phi|_{[a,\, b]}) = \Delta(\Phi|_{[a,\, c]}) + \Delta(\Phi|_{[c,\, b]}).$$

From conjugation invariance of ρ, we see that $\Delta(\Psi^{-1}\Phi\Psi) = \Delta(\Phi)$ for any two continuous paths Φ and Ψ in $\mathrm{Sp}(V)$. Moreover, when $B\colon W \to V$ is a symplectic transformation,

$$\Delta(B^{-1}\Phi B) = \Delta(\Phi). \tag{2.1}$$

Finally, assume that $\Phi(0) = I$ and $\Phi(T) - I$ is non-degenerate. (Here $\Phi\colon [0, T] \to \mathrm{Sp}(V)$.) Then the *Conley–Zehnder index* $\mu_{\mathrm{CZ}}(\Phi)$ is defined (see [CZ2] and also, e.g., [Sa, SZ]) and, as is shown in [SZ],

$$|\,\mu_{\mathrm{CZ}}(\Phi) - \Delta(\Phi)| \leq \dim V/2. \tag{2.2}$$

We refer the reader to [SZ] for proofs of these facts and for a more detailed discussion of the invariant Δ.

2.2.2. *The quasi-morphism property.* One additional property of Δ important for the proof of Theorem 1.1 is that $\Delta\colon \widetilde{\mathrm{Sp}}(V) \to \mathbb{R}$ is a *quasi-morphism*, i.e., for any two elements Φ and Ψ in $\widetilde{\mathrm{Sp}}(V)$, we have

$$|\Delta(\Psi\Phi) - \Delta(\Psi) - \Delta(\Phi)| \leq C, \tag{2.3}$$

where the constant $C \geq 0$ is independent of Ψ and Φ.

To simplify the notation, throughout the rest of this section we will denote by C a positive constant depending only on $\dim V$ – as is the case in (2.3). However, C may assume different values in different formulas.

With this convention in mind, (2.3) is easily seen to be equivalent to that

$$|\Delta(A\Phi) - \Delta(\Phi)| \leq C \tag{2.4}$$

for any continuous path Φ in $\mathrm{Sp}(V)$, not necessarily originating at the identity, and for any $A \in \mathrm{Sp}(V)$.

The quasi-morphism property (2.3) is well known to hold for several other maps $\widetilde{\mathrm{Sp}}(V) \to \mathbb{R}$ which are similar to Δ (see [BG]) and can be established for Δ in a number of ways as a consequence of the quasi-morphism property for one of these maps.

For instance, recall that every $A \in \mathrm{Sp}(V)$ can be uniquely represented as a product $A = QU$, where U is unitary (with respect to a fixed, compatible with ω complex structure) and Q is symmetric and positive definite. (This is the so-called *polar decomposition*.) Set $\tilde{\rho}(A) = \det_{\mathbb{C}} U$ and define $\tilde{\Delta}$ in the same way as Δ, but with $\tilde{\rho}$ in place of ρ. (In contrast with ρ and Δ, the maps $\tilde{\rho}$ and $\tilde{\Delta}$ depend on the choice of complex structure.) It is known that the map $\tilde{\Delta}\colon \widetilde{\mathrm{Sp}}(V) \to \mathbb{R}$ is a quasi-morphism; see [Du] and also [BG] for further references. Furthermore, as is shown in [BG, Section C-2], $\Delta(\Phi) = \lim_{k\to\infty} \tilde{\Delta}(\Phi^k)/k$ for $\Phi \in \widetilde{\mathrm{Sp}}(V)$. Now it is easy to see that (2.3) holds for Δ since it holds for $\tilde{\Delta}$.

Remark 2.1. Alternatively, to prove (2.3), one can first show that $|\tilde{\Delta} - \Delta| \leq C$ on $\widetilde{\mathrm{Sp}}(V)$ and then use again the fact that $\tilde{\Delta}$ is a quasi-morphism. (This argument was communicated to us by M. Entov and L. Polterovich, [EP1].) In fact, once the inequality $|\tilde{\Delta}-\Delta| \leq C$ is established, it is not hard to prove directly that both maps Δ and $\tilde{\Delta}$ are quasi-morphisms by using the polar decomposition and "alternating" between these two maps. The only step which is, perhaps, not immediate is that (2.4) holds for Δ when A and Φ are both symmetric and positive definite. This, however, follows from the elementary fact that in this case the eigenvalues of $A\Phi(t)$ are real for all t (even though $A\Phi(t)$ is not necessarily symmetric), and hence $\Delta(A\Phi) = \Delta(\Phi) = 0$.

Remark 2.2. It is worth mentioning that any of Maslov type quasi-morphisms on $\widetilde{\mathrm{Sp}}(V)$ (see, e.g., [BG, EP2, RS, SZ]) can be used in the proof of Theorem 1.1. The only features of a quasi-morphism essential for the argument are the normalization (behavior on $\mathrm{U}(V)$) and the Sturm comparison theorem (Proposition 2.3 below). The latter obviously holds for any of these quasi-morphisms, once it is established for one, for the difference between any two of such quasi-morphisms is bounded. The properties that set Δ apart from other quasi-morphisms are that Δ is continuous and conjugation invariant and homogeneous (i.e., $\Delta(\Phi^k) = k\Delta(\Phi)$; see [SZ]). These facts, although used in the proof for the sake of simplicity, are not really crucial for the argument.

2.3. **Sturm comparison theorem.** A time-dependent, quadratic Hamiltonian $H(t)$ on (V, ω) generates a linear time-dependent flow $\Phi_H(t) \in \mathrm{Sp}(V)$ via the Hamilton equation. Once V is identified with $\mathbb{R}^{2n} = \mathbb{C}^n$, this equation takes the form

$$\dot{\Phi}_H = JH(t)\Phi_H(t),$$

where J is the standard complex structure. We say that $H_1 \geq H_0$ when $H_1 - H_0$ is positive semi-definite, i.e., $H_1 - H_0$ is a non-negative function on V. Likewise, we write $H_1 - H_0 > 0$ if $H_1 - H_0$ is positive definite.

Proposition 2.3 (Sturm Comparison Theorem)**.** *Assume that $H_1 \geq H_0$ for all t. Then*

$$\Delta(\Phi_{H_1}) \geq \Delta(\Phi_{H_0}) - C$$

as functions of t.

This result is yet another version of the comparison theorem in (symplectic) Sturm theory, similar to those established in, e.g., [Ar1, Bo, Ed]. The proposition can be easily verified by combining the construction of the generalized Maslov index, [RS], with the Arnold comparison theorem, [Ar1], and utilizing (2.2). For the sake of completeness, we give a detailed proof.

Proof. Due to continuity of Δ, by perturbing H_1 and H_0 if necessary, we may assume without loss of generality that $H_1 - H_0 > 0$ for all t. Furthermore, by the quasi-morphism property (2.4), we may also assume that $\Phi_{H_0}(0) = \Phi_{H_1}(0)$.

Set $H_s = (1-s)H_0 + sH_1$ and let $\Phi_s(t)$ stand for the flow of H_s with the initial condition $\Phi_s(0)$ independent of s. Thus

$$\dot{\Phi}_s = JH_s\Phi_s. \tag{2.5}$$

Fix $T > 0$. The path $\Phi_1(t)$ with $t \in [0, T]$ is homotopic to the concatenation of $\Phi_0(t)$ and the path $\Psi(s) = \Phi_s(T)$, $s \in [0, 1]$. Hence, it suffices to show that

$$\Delta(\Psi) \geq -C. \tag{2.6}$$

Denote by $K_s(t)$ the quadratic Hamiltonian generating the family $s \mapsto \Phi_s(t)$ for a fixed time $t \in [0, T]$. To establish (2.6), let us first show that $K_s(T) > 0$ for all $s \in [0, 1]$. Using continuity of Δ as above, we may assume without loss of generality that $K_s(t)$ degenerates only for a finite collection of points $0 = t_0 < t_1 < \ldots < t_k < T$. It is well known that the positive inertia index of $K_s(t)$ increases as t goes through t_i provided that the restriction of $\dot{K}_s(t_i)$ to $\ker K_s(t_i)$ is positive definite; see e.g., [Ar1]. Linearizing the Hamilton equation (2.5) with respect to s, we obtain by a simple calculation that

$$\dot{K}_s = \dot{H}_s + \{K_s, H_s\},$$

where $\{K_s, H_s\} = H_sJK_s - K_sJH_s$ (the Poisson bracket). Note that $\{K_s, H_s\}(x) = -2\langle K_sx, JH_sx\rangle$. Hence, $\{K_s, H_s\}(t_i)$ vanishes on $\ker K_s(t_i)$. Furthermore, $\dot{H}_s = H_1 - H_0 > 0$ on V and, as a consequence, $\dot{K}_s(t_i)$ is positive definite on $\ker K_s(t_i)$. Finally, $K_s(0) = 0$, for $\Phi_s(0)$ is independent of s, and we conclude that $K_s(t) > 0$ for all $s \in [0, 1]$ and all $t \in (0, T]$ and, in particular, for $t = T$.

Returning to the proof of (2.6), set $\tilde{\Psi}(s) = \Psi(s)\Psi(0)^{-1}$. This family is again generated by $K_s(T)$, but now the initial condition is $\tilde{\Psi}(0) = I$. Due to the quasi-morphism property (2.4), it suffices to prove that $\Delta(\tilde{\Psi}) \geq -C$. We will show that $\Delta(\tilde{\Psi}) \geq 0$. As above, by continuity, we may assume that $I - \tilde{\Psi}(s)$ degenerates only for a finite collection of points $0 = s_0 < s_1 < \ldots < s_l < 1$. (In particular, $I - \tilde{\Psi}(1)$ is non-degenerate.) Then $\mu_{\rm CZ}(\tilde{\Psi})$ is defined and, as is proved in [RS],

$$\mu_{\rm CZ}(\tilde{\Psi}) = \frac{1}{2}\,\mathrm{sign}(K_0(T)) + \sum_i \mathrm{sign}\left(K_{s_i}(T)|_{V_i}\right),$$

where $V_i = \ker(I - \tilde{\Psi}_{s_i}(T))$ and sign denotes the signature of a quadratic form. Since, $K_s(T) > 0$ for all s, we see that $\mu_{\mathrm{CZ}}(\tilde{\Psi}) \geq n$ and, by (2.2), $\Delta(\tilde{\Psi}) \geq 0$. This completes the proof of (2.6) and the proof of the proposition. □

Example 2.4. Let $H(t)$ be a quadratic Hamiltonian on $\mathbb{R}^{2n}$ such that $H(t)(X) \geq \alpha\|X\|^2$ for all t, where $\|X\|$ stands for the standard Euclidean norm of $X \in \mathbb{R}^{2n}$ and α is a constant. Then, for all t,

$$\Delta(\Phi_H) \geq 2n\alpha \cdot t - C.$$

More generally, let $H(t)$ be a quadratic Hamiltonian on $\mathbb{R}^{2n_1} \times \mathbb{R}^{2n_2}$ such that $H(t)((X,Y)) \geq \alpha\|X\|^2 - \beta\|Y\|^2$ for all t, where $X \in \mathbb{R}^{2n_1}$ and $Y \in \mathbb{R}^{2n_2}$ and α and β are constants. Then

$$\Delta(\Phi_H) \geq 2(n_1\alpha - n_2\beta)t - C.$$

These inequalities readily follow from Proposition 2.3 by a direct calculation.

2.4. **The Salamon–Zehnder invariant for integral curves.**

2.4.1. *Definitions.* Let $\gamma\colon [0, T] \to P$ be an integral curve of the Hamiltonian flow φ_H^t of a time-dependent Hamiltonian $H = H_t$ on a symplectic manifold P. Let also ξ be a symplectic trivialization of TP along γ, i.e., $\xi(t)$ is a symplectic basis in $T_{\gamma(t)}P$ depending smoothly or continuously on t. The trivialization ξ gives rise to a symplectic identification of the tangent spaces $T_{\gamma(t)}P$ with $T_{\gamma(0)}P$, and hence the linearization of φ_H^t along γ can be viewed as a family $\Phi(t) \in \mathrm{Sp}(T_{\gamma(0)}P)$. We set $\Delta_\xi(\gamma) := \Delta(\Phi)$. This is the Salamon–Zehnder invariant of γ with respect to ξ. Clearly, $\Delta_\xi(\gamma)$ depends on ξ.

Assume now that γ is a contractible T-periodic orbit of H. Recall that a *capping* of γ is an extension of γ to a map $v\colon D^2 \to P$. A capping gives rise to a symplectic trivialization of TP along v and hence along γ, unique up to homotopy, and we denote by $\Delta_v(\gamma)$ the Salamon–Zehnder invariant of γ evaluated with respect to this trivialization. Note that $\Delta_v(\gamma)$ is determined entirely by the homotopy class of v and it is well known that adding a sphere $w \in \pi_2(P)$ to v results in the Salamon–Zehnder invariant changing by $-2\int_w c_1(TP)$. In particular, $\Delta(\gamma) := \Delta_v(\gamma)$ is independent of v whenever $c_1(TP)|_{\pi_2(P)} = 0$.

When γ is *non-degenerate*, i.e., $d\varphi_H^T\colon T_{\gamma(0)}P \to T_{\gamma(0)}P$ does not have one as an eigenvalue, the Conley–Zehnder index $\mu_{\mathrm{CZ}}(\gamma)$ is defined as $\mu_{\mathrm{CZ}}(\Phi)$ in the same way as $\Delta(\gamma)$ by using a trivialization along γ; see [CZ2, Sa, SZ]. Then inequality (2.2) relating Δ and μ_{CZ} turns into

$$|\,\mu_{\mathrm{CZ}}(\gamma) - \Delta(\gamma)| \leq \dim P/2. \tag{2.7}$$

Note that in general $\mu_{\mathrm{CZ}}(\gamma)$ depends on the choice of trivialization along γ. Thus, in (2.7) we assumed that both invariants are taken with respect to the same trivialization, e.g., with respect to the same capping, unless $c_1(TP)|_{\pi_2(P)} = 0$ and the choice of capping is immaterial for either invariant; see, e.g., [Sa]. When the choice of capping v is essential, we will use the notation $\Delta_v(\gamma)$ and $\mu_{\mathrm{CZ}}(\gamma, v)$.

Example 2.5. Let $K\colon \mathbb{R}^{2n} \to \mathbb{R}$ be a convex autonomous Hamiltonian such that $d^2K \geq \alpha \cdot I$ at all points, where α is a constant. Then, as is easy to see from Example 2.4, $\Delta(\gamma) \geq 2n\alpha \cdot T - C$ for any integral curve $\gamma\colon [0, T] \to \mathbb{R}^{2n}$. Note that here $\Delta(\gamma)$ is evaluated with respect to the standard Euclidean trivialization and we are not assuming that the curve γ is closed.

2.4.2. *Change of the Hamiltonian.* Consider two autonomous Hamiltonians H and K on a symplectic manifold P such that H is an increasing function of K, i.e., $H = f \circ K$, where $f \colon \mathbb{R} \to \mathbb{R}$ is an increasing function. Let γ be a periodic orbit of K lying on an energy level, which is regular for both K and H. Then γ can also be viewed, up to a change of time, as a periodic orbit of H. Fixing a trivialization of TP along γ, we have the Salamon–Zehnder invariants, $\Delta(\gamma, K)$ and $\Delta(\gamma, H)$ of γ defined for the flows of K and H. The following result, used in the proof of Theorem 1.1, is nearly obvious:

Lemma 2.6. *Under the above assumptions, $\Delta(\gamma, K) = \Delta(\gamma, H)$.*

Proof. Set $H_s = (1-s)K + sH$, where $s \in [0, 1]$. These Hamiltonians are functions of $H_0 = K$ and the level containing γ is regular for each H_s. Furthermore, after multiplying K and H by positive constants, we may assume that γ has period equal to one for all H_s. Denote by $\Phi_s(t)$ the linearization of the flow $\varphi^t_{H_s}$ of H_s along γ interpreted, using the trivialization, as a path in $\mathrm{Sp}(T_zP)$, where $z = \gamma(0)$. Clearly,

$$\Delta(\gamma, K) = \Delta(\Phi_0) \text{ and } \Delta(\gamma, H) = \Delta(\Phi_1).$$

The path $\Phi_1(t)$ is homotopic to the concatenation of $\Phi_0(t)$ and the path $\Psi(s) = \Phi_s(1)$. Hence,

$$\Delta(\Phi_1) = \Delta(\Phi_0) + \Delta(\Psi),$$

and it is sufficient to show that $\Delta(\Psi) = 0$. To this end, we will prove that all maps $\Psi(s) = (d\varphi^1_{H_s})_z \colon T_zP \to T_zP$ have the same eigenvalues.

Note that for all s the maps $\Psi(s)$ are symplectic and preserve the hyperplane E tangent to the energy level through z. The eigenvalues of $\Psi(s)$ are those of $\Psi(s)|_E$ and the eigenvalue one corresponding to the normal direction to E. Furthermore, all maps Ψ also preserve the one-dimensional space E^ω spanned by $\gamma'(0)$ and are equal to the identity on this space. The quotient E/E^ω can be identified with the space normal to $\gamma'(0)$ in E and the map $\bar{\Psi}(s) \colon E/E^\omega \to E/E^\omega$ induced by $\Psi(s)|_E$ is the linearized return map along γ in the energy level containing γ. Thus, this map is independent of s. As a consequence, the maps $\Psi(s)|_E$, and hence $\Psi(s)$, have the same eigenvalues for all $s \in [0, 1]$. □

3. Sturm comparison theorems for periodic orbits near Morse–Bott non-degenerate symplectic extrema

3.1. **Growth of Δ.** Let, as in Theorem 1.1, $K \colon P \to \mathbb{R}$ be an autonomous Hamiltonian attaining its Morse-Bott non-degenerate minimum $K = 0$ along a closed symplectic submanifold $M \subset P$. The key to the proof of Theorem 1.1 is the following result, generalizing Example 2.5, which is essentially a version of the Sturm comparison theorem for K:

Proposition 3.1. *Assume that $c_1(TP) = 0$. Then there exist constants $a > 0$ and c and $r_0 > 0$ such that, whenever $0 < r < r_0$,*

$$\Delta(\gamma) \geq a \cdot T - c \tag{3.1}$$

for every contractible T-periodic orbit γ of K on the level $K = r^2$.

Along with this proposition, we also establish a lower bound on $\Delta(\gamma)$ that holds without the assumption that $c_1(TP) = 0$. Fix a closed 2-form σ with $[\sigma] = c_1(TP)$. For instance, we can take as σ the Chern–Weil form representing c_1 with respect to a Hermitian connection on TP. In the notation of Section 2.4.1, we have

Proposition 3.2. *There exist constants $a > 0$ and c and $r_0 > 0$ such that, whenever $0 < r < r_0$,*

$$\Delta_v(\gamma) \geq a \cdot T - c - 2 \int_v \sigma \tag{3.2}$$

for every contractible T-periodic orbit γ of K on the level $K = r^2$ with capping v.

3.2. **Proof of Propositions 3.1 and 3.2.** The idea of the proof is that the fiber contribution to $\Delta(\gamma)$ is of order T and positive, while the base contribution is of order $r \cdot T$. It will be convenient to prove a superficially more general form of (3.1) and (3.2). Namely, we will show that

$$\Delta(\gamma) \geq (a - b \cdot r)T - c \tag{3.3}$$

and

$$\Delta_v(\gamma) \geq (a - b \cdot r)T - c - 2 \int_v \sigma \tag{3.4}$$

for some constants $a > 0$ and b and c, when $r > 0$ is small. This implies (3.1) and (3.2) with perhaps a slightly smaller value of a.

Throughout the rest of this section we adopt the following notational convention: in all expressions *const* stands for a constant which is independent of r and γ and T, once r is sufficiently small. The value of this constant (immaterial for the proof) is allowed to vary from one formula to another. A similar convention is also applied to the constants $a > 0$ and b and c.

3.2.1. *Particular case: an integral curve in a Darboux chart.* Before turning to the general case, let us prove (3.3) for an integral curve γ of K contained in a Darboux chart. Let $U \subset M$ be a contractible Darboux chart. The inclusion $U \hookrightarrow M$ can be extended to a symplectic embedding of an open set $U \times V \hookrightarrow P$, where V is a ball (centered at the origin) in a symplectic vector space and $U \times V$ carries the product symplectic structure. In what follows, we identify $U \times V$ with its image in P and U with $U \times 0$. Note that then $T_{(x,0)}(x \times V)$, where $x \in U$, is the symplectic orthogonal complement $(T_xM)^\omega$ to T_xM.

Let $\gamma\colon [0, T] \to U \times V$ be an integral curve of the flow of K on an energy level $K = r^2$. We emphasize that at this stage we do not require γ to be closed, but we do require it to be entirely contained in $U \times V$. The coordinate system in $U \times V$ gives rise to a symplectic trivialization of TP along γ and we denote by $\Delta(\gamma)$ the Salamon–Zehnder invariant of the linearized flow along γ with respect to this trivialization; see Section 2.4.1.

Next we claim that (3.3) *holds for such an integral curve γ with all constants independent of γ.*

Indeed, the linearized flow of K along γ is given by the quadratic Hamiltonian equal to the Hessian $d^2K_{\gamma(t)}$ evaluated with respect to the coordinate system. On the other hand, since d^2K is positive definite in the direction normal to the critical manifold M, we have

$$d^2K_{(x,y)}(X, Y) \geq a\|Y\|^2 - b \cdot r\|X\|^2. \tag{3.5}$$

Here $(x, y) \in U \times V$ and $X \in T_xU$ and $Y \in T_yV$ and $r^2 = K(x, y)$. Note that the constants $a > 0$ and b depend on K and the coordinate chart $U \times V$, but not on γ and r. The lower bound (3.3) (with values of a and b different from those in (3.5)) follows now from the comparison theorem (Proposition 2.3) and Example 2.4; cf. Example 2.5.

3.2.2. *Length estimate.* Fix an almost complex structure J on P compatible with ω and such that M is an almost complex submanifold of P, i.e., $J(TM) = TM$. The pair J and ω gives rise to a Hermitian metric on the complex vector bundle $TP \to P$. We denote by $l(\gamma)$ the length of a smooth curve γ in P with respect to this metric. Furthermore, there exists a unique Hermitian connection on TP, i.e., a unique connection such that parallel transport preserves the metric and J, and hence, ω. (Note that, unless J is integrable, this connection is different from the Levi–Civita connection.)

Let $\gamma\colon [0, T] \to P$ be an integral curve of K (not necessarily closed) on the level $K = r^2$. Then, since M is a critical manifold of K, we have

$$l(\gamma) \leq const \cdot r \cdot T. \tag{3.6}$$

As the first application of (3.6), observe that Proposition 3.1 is a consequence of Proposition 3.2, i.e., (3.4) implies (3.3). Indeed, assume that $c_1(TP) = 0$, i.e., $\sigma = d\alpha$ for some 1-form α on P. Then, by Stokes' formula and (3.6),

$$\left| \int_v \sigma \right| = \left| \int_\gamma \alpha \right| \leq const \cdot \|\alpha\|_{C^0} \cdot r \cdot T,$$

which, combined with (3.4), implies (3.3).

Before proceeding with a detailed proof of (3.4), let us briefly outline the argument. We will cover a closed T-periodic orbit γ of K on the level $K = r^2$ by a finite collection of Darboux charts. The required number N of charts is of order $l(\gamma) \sim r \cdot T$. Within every chart, as was proved in Section 3.2.1, we have a lower bound on Δ with respect to the Euclidean trivialization. Combined, these trivializations can be viewed as an approximation to a Hermitian-parallel trivialization ξ along $\gamma\colon [0, T] \to P$. (We do not assume that $\xi(0) = \xi(T)$.) Furthermore, within every chart the discrepancy between Salamon–Zehnder invariants for the two trivializations (Euclidean and Hermitian-parallel) is bounded by a constant independent of γ and r. As a consequence, the difference between $\Delta_\xi(\gamma)$ and the total Salamon–Zehnder invariant for Euclidean chart-wise trivializations is of order $N \sim r \cdot T$, and we conclude that (3.3) holds for $\Delta_\xi(\gamma)$. Finally, by the Gauss–Bonnet theorem, the effect of replacing ξ by a trivialization associated with a capping is captured by the integral term in (3.4).

3.2.3. *Auxiliary structure: a Darboux family.* To introduce a Darboux family in P along M, let us first set some notation. Denote by $B_x(\delta) \subset T_xM$ and $B_x^\perp(\delta^\perp) \subset (T_xM)^\omega$ the balls of radii $\delta > 0$ and $\delta^\perp > 0$, respectively, centered at the origin and equipped with the symplectic structures inherited from T_xM and $(T_xM)^\omega$.

The first component of the Darboux family is a symplectic tubular neighborhood $\pi\colon W \to M$. This is an ordinary tubular neighborhood of M, i.e., an identification of a neighborhood W of M in P with a neighborhood of the zero section in $(TM)^\omega = TM^\perp$ formed by the fiber-wise balls $B_x^\perp(\delta^\perp)$, such that the diffeomorphisms between the fibers $V_x = \pi^{-1}(x)$ and the balls $B_x^\perp(\delta^\perp)$ preserve the symplectic structure. In particular, we obtain a family of symplectic embeddings $B_x^\perp(\delta^\perp) \to P$ sending the origin to x and depending smoothly on x. The linearization of the map $B_x^\perp(\delta^\perp) \to V_x$ at x is the inclusion $(T_xM)^\omega \hookrightarrow T_xP$.

The second component is a Darboux family in M. This is a family of symplectic embeddings $T_xP \supset B_x(\delta) \to M$ depending smoothly on $x \in M$, sending the origin $0 \in T_xM$ to x, and having the identity linearization at $0 \in T_xM$. It is easy to

see that such a Darboux family exists provided that $\delta > 0$ is sufficiently small; see [We2]. We denote the images of this embedding by $U_x \subset M$.

Now we extend each pair of symplectic embeddings $B_x^{\perp}(\delta^{\perp}) \to P$ and $B_x(\delta) \to M$ to a symplectic embedding $T_xP \supset B_x(\delta) \times B_x^{\perp}(\delta^{\perp}) \to P$, which is again required to depend smoothly on $x \in M$. The resulting maps will be called a *Darboux family (in P along M)*.

Let W_x stand for the image of the embedding $B_x(\delta) \times B_x^{\perp}(\delta^{\perp}) \to P$. Note that W_x is naturally symplectomorphic to $U_x \times V_x$ with the split symplectic structure and the tangent space to $y \times V_x$ is $(T_yM)^{\omega}$ for every $y \in U_x$. We also denote by $\pi_x \colon W_x \to U_x$ the projection to the first factor. (At this point it is worth emphasizing that in general π and π_x do not agree on W_x although $\pi(V_x) = x = \pi_x(V_x)$.) Whenever the values of radii δ and $\delta^{\perp}$ are essential, we will use the notation $U_x(\delta)$ and $V_x(\delta^{\perp})$ and $W_x(\delta, \delta^{\perp})$ and $W(\delta^{\perp})$. Henceforth, we fix a Darboux family with some $\delta_0 > 0$ and $\delta_0^{\perp} > 0$ and consider only Darboux families obtained by restricting the fixed one to smaller balls.

Let us now state a few simple properties of Darboux families, which are used in the rest of the proof. These properties require $\delta > 0$ and $\delta^{\perp} > 0$ to be sufficiently small. However, once this is the case, all constants involved are independent of δ and $\delta^{\perp}$.

(DF1) The Euclidean metric on W_x, arising from the Darboux diffeomorphism of W_x with an open subset of T_xP, is equivalent to the restriction of the Hermitian metric to W_x. Moreover, the constants involved can be taken independent of x.

As a consequence of this obvious observation we need not distinguish between the Hermitian and Euclidean metric on W_x in (DF2) and (DF3) below.

(DF2) The inequality (3.5) holds in each chart W_x with some constants $a > 0$ and b independent of x.

(DF3) The difference between Euclidean and Hermitian parallel transports along any short curve contained in W_x is small for all $x \in M$. More specifically, denote by Π_η^E and Π_η^H the Euclidean and Hermitian parallel transports $T_{\eta(0)}P \to T_{\eta(1)}P$ along a curve $\eta \colon [0, 1] \to W_x$. For any $\epsilon > 0$ there exists l_0, depending on ϵ but not on δ and $\delta^{\perp}$, such that for any $x \in M$ and any curve η in W_x with $l(\eta) \leq l_0$, the symplectic transformation $(\Pi_\eta^H)^{-1}\Pi_\eta^E$ lies in the ϵ-neighborhood of the identity in $\mathrm{Sp}(T_{\eta(0)}P)$.

The property (DF2) is a consequence of the fact that the linearization of a Darboux map $B_x \times B_x^{\perp} \to W_x$ at the origin is the identity map on T_xP. Assertion (DF3) is established by the standard argument.

Now we fix a small $\epsilon > 0$ and $\delta > 0$ and $\delta^{\perp} > 0$ such that (DF1) and (DF2) hold and the distance from $V_x(\delta^{\perp}/2)$ to the boundary of $W_x = W_x(\delta, \delta^{\perp})$ is smaller than $l_0(\epsilon)$. This is possible since l_0 is independent of δ and $\delta^{\perp}$.

Remark 3.3. In fact, $\epsilon > 0$ need not be particularly small. It suffices to ensure that the value of the Salamon–Zehnder invariant Δ on any path in the ϵ-neighborhood of the identity is bounded by a constant independent of the path. This is always the case when the neighborhood is simply connected (and has compact closure).

3.2.4. *Proof of* (3.4). Let $r > 0$ be so small that the level $K = r^2$ is entirely contained in the tubular neighborhood $W(\delta^{\perp}/2)$. Then this level is also contained

in the union of the charts $W_x(\delta, \delta^\perp/2)$ and hence in the union of the charts W_x. Let $\gamma\colon [0,\, T] \to P$ be a T-periodic orbit of K on the level.

Fix a unitary frame $\xi(0)$ in $T_{\gamma(0)}P$ and extend this frame to a Hermitian trivialization ξ of TP along the path γ by applying Hermitian parallel transport to $\xi(0)$. Note that the resulting trivialization need not be a genuine trivialization along γ viewed as a closed curve: $\xi(0) \neq \xi(T)$. Nonetheless, the Salamon–Zehnder invariant $\Delta_\xi(\gamma)$ of γ with respect to ξ is obviously defined. Namely, recall from Section 2.4.1 that using ξ we can view the linearized flow along γ as a family $\Phi(t) \in \mathrm{Sp}(T_{\gamma(0)}P)$. Then $\Delta_\xi(\gamma) = \Delta(\Phi)$. Our first objective is to show that (3.3) holds for $\Delta_\xi(\gamma)$, i.e.,

$$\Delta_\xi(\gamma) \geq (a - b\cdot r)T - c, \tag{3.7}$$

where the constants $a > 0$ and b and c are independent of r and T and γ.

To this end, consider the partition of $I = [0,\, T]$ into intervals $I_j = [t_{j-1},\, t_j]$ with $j = 1, \ldots, N$ by points

$$0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = T$$

such that the length of $\gamma_j = \gamma|_{I_j}$ is exactly l_0. (The last segment γ_N may have length smaller than l_0.) It is essential for what follows that, by (3.6),

$$N \leq 1 + const \cdot r \cdot T. \tag{3.8}$$

(Note that, in contrast with the curves γ_j, the intervals I_j are not necessarily short: the average length of I_j is $T/N \sim 1/r$.) Let τ_j be the middle point of I_j, i.e., $\tau_j = (t_{j-1} + t_j)/2$, and $z_j = \gamma(\tau_j)$ and $x_j = \pi(z_j)$. Due to our choice of r, we have $z_j \in V_{x_j}(\delta^\perp/2)$, and, by the choice of δ and $\delta^\perp$, the path γ_j lies entirely in W_{x_j}. We denote by $\Phi|_{I_j}$ the restriction of the family $\Phi(t)$ to I_j. Thus,

$$\Delta_\xi(\gamma) = \sum \Delta(\Phi|_{I_j}). \tag{3.9}$$

We bound $\Delta(\Phi|_{I_j})$ from below in a few steps. First, consider the family $\Phi_j(t) \in \mathrm{Sp}(T_{z_j}P)$ parametrized by $t \in I_j$ and obtained from the linearized flow of K along γ_j by identifying $T_{\gamma(t)}P$ with $T_{z_j}P$ via Hermitian parallel transport. It is easy to see that

$$\Phi_j(t) = \Pi_j \Phi(t)\Phi(\tau_j)^{-1}\Pi_j^{-1},$$

where $\Pi_j\colon T_{z_0}P \to T_{z_j}P$ is the Hermitian parallel transport along γ. By conjugation invariance of Δ (see (2.1)) and the quasi-morphism property (2.4),

$$\Delta(\Phi|_{I_j}) \geq \Delta(\Phi_j) - const, \tag{3.10}$$

where the constant depends only on $\dim P$. Furthermore, let $\Psi_j(t)$ be defined similarly to $\Phi_j(t)$, but this time making use of Euclidean parallel transport in W_{z_j}. Clearly,

$$\Psi_j(t) = A_j(t)\Phi_j(t),$$

where $A_j(t) \in \mathrm{Sp}(T_{z_j}P)$ measures the difference between the Hermitian and Euclidean parallel transports along γ_j. Since $l(\gamma_j) \leq l_0$, we infer from (DF3) that $A_j(t)$ lies in the ϵ-neighborhood of the identity and thus $\Delta(A_j) \leq const$, where the constant is independent of j and γ and r; see Remark 3.3. Due to the quasi-morphism property (2.3) of Δ, we have

$$\Delta(\Phi_j) \geq \Delta(\Psi_j) - \Delta(A_j) - const \geq \Delta(\Psi_j) - const. \tag{3.11}$$

By (DF2), the argument from Section 3.2.1 applies to Ψ_j, and hence

$$\Delta(\Psi_j) \geq (a - b\cdot r)(t_j - t_{j-1}) - const. \tag{3.12}$$

Combining (3.10)–(3.12), we see that

$$\Delta(\Phi|_{I_j}) \geq (a - b \cdot r)(t_j - t_{j-1}) - const, \tag{3.13}$$

where all constants are independent of γ and r and the chart, and $a > 0$.

Finally, adding up inequalities (3.13) for all $j = 1, \dots, N$ and using (3.9), we obtain

$$\Delta(\Phi) \geq (a - b \cdot r)T - const \cdot N,$$

which in conjunction with (3.8) implies (3.7).

To finish the proof of (3.4), fix a capping v of γ and let ζ be a Hermitian trivialization of TP along γ associated with v. Identifying the spaces $T_{\gamma(t)}P$ via ζ, we can view the linearized flow of K along γ as a family $\tilde{\Phi}(t) \in \mathrm{Sp}(T_{z_0}P)$, $t \in I$. By definition, $\Delta_v(\gamma) = \Delta(\tilde{\Phi})$. Furthermore, without loss of generality we may assume that $\zeta(0) = \xi(0)$ and then

$$\tilde{\Phi}(t) = B(t)\Phi(t).$$

Here the transformations $B(t) \in \mathrm{U}(T_{z_0}P)$ send the frame $\zeta(0)$ to the frame $\xi(t)$, where the latter is regarded as a frame in $T_{z_0}P$ by means of ζ. Due to again the quasi-morphism property,

$$\Delta(\tilde{\Phi}) \geq \Delta(\Phi) + \Delta(B) - const.$$

Since the transformations $B(t)$ are unitary, $\rho(B(t)) = \det_{\mathbb{C}} B(t)$, and $\Delta(B)$ is the "total rotation" of $\det^2_{\mathbb{C}} B$. Hence, by the Gauss–Bonnet theorem,

$$\Delta(B) = -2 \int_v \sigma,$$

where σ is the Chern–Weil form representing $c_1(TP)$. Combined with (3.7), this concludes the proof of (3.4) and the proof of Propositions 3.1 and 3.2.

4. Particular case: P is geometrically bounded and symplectically aspherical

To set the stage for the proof of the general case, in this section we establish Theorem 1.1 under the additional assumptions that P is geometrically bounded and symplectically aspherical (i.e., $\omega|_{\pi_2(P)} = 0 = c_1(TP)|_{\pi_2(P)}$). We refer the reader to, e.g., [AL, CGK, GG2] for the definition and a detailed discussion of geometrically bounded manifolds. Here we only mention that among such manifolds are all closed symplectic manifolds as well as their covering spaces, manifolds that are convex at infinity (e.g., $\mathbb{R}^{2n}$, cotangent bundles, and symplectic Stein manifolds) and also twisted cotangent bundles.

4.1. **Conventions.** Throughout the rest of the paper we adopt the following conventions and notation. Let $\gamma\colon S^1_T \to P$, where $S^1_T = \mathbb{R}/T\mathbb{Z}$, be a contractible loop with capping v. The action of a T-periodic Hamiltonian H on (γ, v) is defined by

$$A_H(\gamma, v) = -\int_v \omega + \int_{S^1_T} H_t(\gamma(t))\, dt,$$

where $H_t = H(t, \cdot)$. When $\omega|_{\pi_2(P)} = 0$, the action $A_H(\gamma, v)$ is independent of the choice of v and we will use the notation $A_H(\gamma)$.

All Hamiltonians considered below are assumed to be one-periodic in time or autonomous. In the former case, we always require T to be an integer; in the latter case, T can be an arbitrary real number.

The least action principle asserts that the critical points of A_H on the space of all (capped) contractible loops $\gamma\colon S^1_T \to P$ are exactly (capped) contractible T-periodic orbits of the time-dependent Hamiltonian flow φ^t_H of H. The Hamiltonian vector field X_H of H, generating this flow, is given by $i_{X_H}\omega = -dH$. The Salamon–Zehnder invariant $\Delta_v(\gamma)$ of a T-periodic orbit γ with capping v and the Conley–Zehnder index $\mu_{\mathrm{CZ}}(\gamma, v)$, when γ is non-degenerate, are defined as in Section 2.4.1 using the linearized flow $d\varphi^t_H$ and a trivialization associated with v.

At this point it is important to emphasize that our present conventions differ from the conventions from, e.g., [Sa], utilized implicitly in Sections 2.4 and 3. For instance, the Hamiltonian vector field X_H defined as above is negative of the Hamiltonian vector field in [Sa]. As a consequence of this sign change, the values of $\Delta_v(\gamma)$ and $\mu_{\mathrm{CZ}}(\gamma, v)$ also change sign. (In other words, from now on the Salamon–Zehnder invariant of a linear flow with positive definite Hamiltonian is negative; equivalently, μ_{CZ} is normalized so that $\mu_{\mathrm{CZ}}(\gamma) = n$ when γ is a non-degenerate maximum of an autonomous Hamiltonian with small Hessian.) In particular, the value of Δ in Propositions 3.1 and 3.2 must in what follows be replaced by $-\Delta$. This change of normalization should not lead to confusion, for the correct sign is always clear from the context, and it will enable us to conveniently eliminate a number of negative signs in the statements of intermediate results.

4.2. **Floer homological counterpart.** The proof of the theorem uses two major ingredients. One is the Sturm comparison theorem for K proved in Section 3. The other is a calculation of the filtered Floer homology for a suitably reparametrized flow of K.

Let, as in Section 3, $K\colon P \to \mathbb{R}$ be an autonomous Hamiltonian attaining its Morse-Bott non-degenerate minimum $K = 0$ along a closed symplectic submanifold $M \subset P$. Pick sufficiently small $r > 0$ and $\epsilon > 0$ with, say, $\epsilon < \epsilon_0 = r^2/10$. Let $H\colon [r^2 - \epsilon,\, r^2 + \epsilon] \to [0, \infty)$ be a smooth decreasing function such that

- $H \equiv \max H$ near $r^2 - \epsilon$ and $H \equiv 0$ near $r^2 + \epsilon$.

Consider now the Hamiltonian equal to $H \circ K$ within the shell bounded by the levels $K = r^2 - \epsilon$ and $K = r^2 + \epsilon$ and extended to the entire manifold P as a locally constant function. Abusing notation, we denote the resulting Hamiltonian by H again. Clearly, $\min H = 0$ on P and the maximum, $\max H$, is attained on the entire domain $K \leq r^2 - \epsilon$.

Proposition 4.1 ([GG2])**.** *Assume that P is geometrically bounded and symplectically aspherical and that $r > 0$ is sufficiently small. Then, once $\max H \geq C(r)$ where $C(r) \to 0$ as $r \to 0$, we have*

$$\mathrm{HF}^{(a,\,b)}_{n_0}(H) \neq 0$$

for $n_0 = 1 + (\operatorname{codim} M - \dim M)/2$ and some interval $(a,\, b)$ with $a > \max H$ and $b < \max H + C(r)$.

Here $\mathrm{HF}^{(a,b)}_*(H)$ stands for the filtered Floer homology of H for the interval (a, b). We refer the reader to Floer's papers [Fl1, Fl2, Fl3, Fl4, Fl5], to, e.g., [BPS, HS, SZ, Sc], or to [HZ3, McSa, Sa] for further references and introductory accounts of the construction of (Hamiltonian) Floer and Floer–Novikov homology. Filtered Floer homology for geometrically bounded manifolds are discussed in detail in, e.g., [CGK, GG2, Gü] and [Gi8] with the above conventions. Finally, the construction of

filtered Floer–Novikov homology for open manifolds, utilized in Section 6, is briefly reviewed in Section 5.

4.3. **Proof of Theorem 1.1: a particular case.** Now we are in a position to prove Theorem 1.1 in the particular case where P is geometrically bounded and symplectically aspherical. First observe that H has a non-trivial contractible one-periodic orbit γ with

$$1 - \dim M = n_0 - \dim P/2 \leq \Delta(\gamma) \leq n_0 + \dim P/2 = 1 + \operatorname{codim} M. \tag{4.1}$$

Indeed, let $\tilde{H}\colon S^1 \times P \to \mathbb{R}$ be a compactly supported, C^1-close to H, non-degenerate perturbation of H. By Proposition 4.1, $\tilde{H}$ has a non-degenerate contractible orbit $\tilde{\gamma}$ with action in the interval (a, b) and Conley–Zehnder index n_0. By (2.7),

$$n_0 - \dim P/2 \leq \Delta(\tilde{\gamma}) \leq n_0 + \dim P/2.$$

Passing to the limit as $\tilde{H} \to H$ and setting $\gamma = \lim \tilde{\gamma}$, we conclude that the same is true for $\Delta(\gamma)$ by continuity of Δ. The orbit γ is non-trivial since the trivial orbits of H have action either zero or $\max H$ while $A_H(\gamma) > a > \max H$. As a consequence, γ lies on a level of H with $r^2 - \epsilon < K < r^2 + \epsilon$.

Since H is a function of K, we may also view γ, keeping the same notation for the orbit, as a T-periodic orbit of K. Note that H is a decreasing function of K, but otherwise the requirements of Lemma 2.6 are met. Hence, $\Delta(\gamma, K) = -\Delta(\gamma)$, where $\Delta(\gamma) = \Delta(\gamma, H)$. Thus (4.1) turns into

$$1 - \dim M \leq -\Delta(\gamma, K) \leq 1 + \operatorname{codim} M.$$

On the other hand, up to a sign, inequality (3.1) of Proposition 3.1 still holds for γ with constants $a > 0$ and c independent of H and r and $\epsilon > 0$:

$$-\Delta(\gamma, K) \geq a \cdot T - c.$$

(The negative sign is a result of the convention change.) Hence, we have an *a priori* bound on T:

$$T \leq T_0 = (1 + c + \operatorname{codim} M)/a.$$

Passing to the limit as $\epsilon \to 0$, we see that the T-periodic orbits γ of K converge, by the Arzela-Ascoli theorem, to a periodic orbit of K on the level $K = r^2$ with period bounded from above by T_0. This completes the proof of Theorem 1.1 in the particular case.

Remark 4.2. In the proof above, the arguments from [CGK, Gü, Ke3] could also be used in place of the result from [GG2]. The only reason for utilizing that particular result is that its proof affords an easy, essentially word-for-word, extension to the general case.

4.4. **Proof of Proposition 1.5.** In the setting of the proposition, fix a trivialization of the normal bundle $(TM)^\omega$. Then, every energy level $K = r^2$ also inherits a trivialization via its identification with the unit sphere bundle in $(TM)^\omega$. When $r > 0$ is small, the field of directions $\ker(\omega|_{K=r^2})$ is transverse to the horizontal distribution. Hence, fixing a horizontal section Γ of the S^1-bundle $\{K = r^2\} \to M$, we obtain the Poincaré return map $\varphi\colon \Gamma \to \Gamma$. Clearly, periodic points of φ are in one-to-one correspondence with periodic orbits of the Hamiltonian flow on $K = r^2$. The restriction $\omega|_\Gamma$ is a symplectic form preserved by φ. Furthermore, as is easy to see, Γ is symplectomorphic to M. Thus, we can view φ as a symplectomorphism

$M \to M$. It is not hard to show that φ is in fact a Hamiltonian diffeomorphism; see [Gi1] and also [Ar2]. The proposition now follows from the Conley conjecture proved in [Gi9].

5. Filtered Floer–Novikov homology for open manifolds

In this section, we describe a version of Floer (or Floer–Novikov) homology which is suitable for extending Proposition 4.1 beyond the class of symplectically aspherical, geometrically bounded manifolds. We will focus on the case where P is open, but not necessarily geometrically bounded, for this is the setting most relevant to the proof of Theorem 1.1. Furthermore, we also assume throughout the construction that P is spherically rational, i.e., $\langle \omega, \pi_2(P)\rangle = \lambda_0\mathbb{Z}$ for some $\lambda_0 > 0$ or $\omega|_{\pi_2(P)} = 0$. In the latter case, it is convenient to set $\lambda_0 = \infty$ and $\lambda_0\mathbb{Z} = \{0\}$.

5.1. **Definitions.** Fix an open set $W \subset P$ with compact closure. Let $H\colon S^1 \times P \to \mathbb{R}$ be a one-periodic Hamiltonian on P, supported in W (or rather in $S^1 \times W$). Two cappings v_0 and v_1 of the same one-periodic orbit γ of H are said to be equivalent if $\langle \omega, w\rangle = 0 = \langle c_1(TP), w\rangle$, where $w \in \pi_2(P)$ is the sphere obtained by attaching v_1 to v_0 along γ. (For instance, when P is symplectically aspherical any two cappings are equivalent.) The value of the action functional $A_H(\gamma, v)$ and the Conley–Zehnder index $\mu_{\scriptscriptstyle CZ}(\gamma, v)$ and the Salamon–Zehnder invariant $\Delta_v(\gamma)$ are entirely determined by the equivalence class of v and from now on we do not distinguish equivalent cappings.

Assume that all one-periodic orbits of H with action in $(0,\, \lambda_0)$ are non-degenerate and that there are only finitely many such orbits. This is a C^∞-generic condition in H, cf. [FHS, HS]. (However, since P is open and H is compactly supported, the flow necessarily has trivial periodic orbits. Such orbits are degenerate and have action in $\lambda_0\mathbb{Z}$.) For $0 < a < b < \lambda_0$ denote by $\mathrm{CF}_k^{(a,\,b)}(H)$ the vector space freely generated over $\mathbb{Z}_2$ by (capped) orbits $x = (\gamma, v)$ with $\mu_{\scriptscriptstyle CZ}(x) = k$ and $a < A_H(x) < b$. Note that each vector space $\mathrm{CF}_k^{(a,\,b)}(H)$ has finite dimension, for changing the equivalence class of a capping by attaching a sphere w to it shifts the action and the index by $\langle \omega, w\rangle \in \lambda_0\mathbb{Z}$ and, respectively, $2\,\langle c_1(TP), w\rangle$.

Fix an almost complex structure J (compatible with ω) on P, which we allow to be time-dependent within W. We define the Floer differential $\partial\colon\ \mathrm{CF}_k^{(a,\,b)}(H) \to \mathrm{CF}_{k-1}^{(a,\,b)}(H)$ by the standard formula:

$$\partial x = \sum \#[\widehat{\mathcal{M}}(x, y)] \cdot y, \tag{5.1}$$

where x is a capped orbit (γ^-, v^-), the sum is taken over all $y = (\gamma^+, v^+)$ with index $k - 1$ and action in $(a,\, b)$, and $\#[\widehat{\mathcal{M}}(x, y)]$ is the number (mod 2) of points in the moduli space $\widehat{\mathcal{M}}(x, y)$ of Floer anti-gradient connecting trajectories from x to y. Let us recall the definition of this moduli space.

Let (s, t) be the coordinates on $\mathbb{R} \times S^1$. Denote by $\mathcal{M}(x, y)$ the space formed by solutions $u\colon \mathbb{R} \times S^1 \to P$ of Floer's equation

$$\frac{\partial u}{\partial s} + J_t(u)\frac{\partial u}{\partial t} = -\nabla H_t(u) \tag{5.2}$$

which are asymptotic to $\gamma^\pm$ at $\pm\infty$, i.e., $u(s, t) \to \gamma^\pm(t)$ point-wise as $s \to \pm\infty$, and such that the capping v^- is equivalent to the one obtained by attaching u to v^+ along γ^+. The space $\mathcal{M}(x, y)$ carries an $\mathbb{R}$-action given by shifts of s. We set

$\widehat{\mathcal{M}}(x, y) = \mathcal{M}(x, y)/\mathbb{R}$. For a generic Hamiltonian H supported in W, the space $\mathcal{M}(x, y)$, equipped with the topology of uniform C^∞-convergence on compact sets, is a smooth manifold of dimension $\mu_{\mathrm{CZ}}(x) - \mu_{\mathrm{CZ}}(y)$; [FHS, HS]. Furthermore, the $\mathbb{R}$-action on $\mathcal{M}(x, y)$ is non-trivial unless $\mathcal{M}(x, y)$ is comprised entirely of one solution $u(s, t)$ independent of s, and thus $\gamma^+ = u = \gamma^-$ and $\dim \mathcal{M}(x, y) = 0$. It follows that $\widehat{\mathcal{M}}(x, y)$ is a discreet set when $\mu_{\mathrm{CZ}}(x) = \mu_{\mathrm{CZ}}(y) + 1$.

As is well known, one cannot expect to have $\partial^2 = 0$ unless P satisfies some additional topological and geometrical requirements. Moreover, the moduli spaces $\widehat{\mathcal{M}}(x, y)$ in (5.1) need not in general be finite and once one of these sets is infinite ∂ is not even defined. The next lemma shows, however, that these problems do not arise if the action interval $(a,\, b)$ is sufficiently short.

Lemma 5.1. *There exists a constant $h = h(W, J) > 0$, depending only on W and J but not on H, such that once $b - a < h$, the zero dimensional moduli spaces $\widehat{\mathcal{M}}(x, y)$ in (5.1) are finite and $\partial^2 = 0$.*

Remark 5.2. As will be clear from the proof of Lemma 5.1, the constant $h > 0$ can be chosen to be the same for all almost complex structures that are sufficiently C^1-close to J. This is essential to ensure generic regularity for Floer continuation maps.

The lemma is nearly obvious. Two phenomena can interfere with compactness of $\widehat{\mathcal{M}}(x, y)$ or cause ∂^2 not to be zero: bubbling-off and the existence of bounded energy sequences of Floer connecting trajectories going to infinity in P. However, since H is supported in W, every Floer connecting trajectory is a holomorphic curve outside W. Leaving a compact neighborhood $\bar{V}$ of W requires such a curve to have energy exceeding some $h_1(V, J) > 0$. Furthermore, since P is spherically rational, bubbling-off requires energy λ_0 or greater. (Alternatively, since Floer trajectories with energy smaller that h_1 are confined to a compact set $\bar{V}$, one can invoke Gromov's compactness theorem rather than rationality of P.) Summarizing, we conclude that neither of these phenomena can occur when $b - a < h = \min\{h_1, \lambda_0\}$. For the sake of completeness, we provide a more detailed argument.

Proof. Fix an open set $V \supset \bar{W}$ with compact closure. Without loss of generality we may assume that $\bar{V}$ and $\bar{W}$ are smooth connected manifolds with boundary. Then $Y = \bar{V} \smallsetminus W$ is a smooth compact domain whose boundary has two components: $\partial\bar{W}$ and $\partial\bar{V}$. There exists a constant $h_1 = h_1(W, V, J) > 0$ such that for every holomorphic curve v in Y whose boundary is contained in ∂Y and meets both of the components of ∂Y we have

$$E(v) := \int_v \omega > h_1.$$

This fact is an immediate consequence of a result of Sikorav, [AL, p. 179]. Namely, consider a holomorphic curve through $z \in P$ with boundary on the R-sphere centered at z. Then according to this result, there exist constants C and $R_0 > 0$ (depending only on z) such that the area of the holomorphic curve is greater than CR^2 whenever $0 < R < R_0$. Moreover, it is clear that C and R_0 can be taken independent of z as long as z varies within a fixed compact set. Let now S be a closed hypersurface in Y separating the two boundary components of ∂Y. Then, any holomorphic curve v as above passes through a point $z \in S$, and thus through a ball of radius $R > 0$, where R depends only on S, centered at z and contained in

Y. Taking $R > 0$ sufficiently small and applying Sikorav's result, we conclude that $E(v) > CR^2 =: h_1$.

Assume now that $b - a < h := \min\{h_1, \lambda_0\}$. Then a Floer trajectory u connecting x to y with $0 < a < A_H(y) \leq A_H(x) < b < \lambda_0$ is necessarily contained in V. Indeed, denote by v a part of u lying in $Y = \bar{V} \smallsetminus W$. Clearly, v is a holomorphic curve (with boundary on ∂Y) since H is supported in W, and

$$E(v) \leq E(u) := \int_{-\infty}^{\infty} \left\| \frac{\partial u}{\partial s} \right\|^2_{L^2(S^1)} ds = A_H(x) - A_H(y) < h_1. \tag{5.3}$$

Furthermore, the boundary of v meets $\partial \bar{W}$, for both x and y are in W, and is entirely contained in $\partial \bar{W}$ since $E(v) < h_1$. Hence, u takes values in V. Thus we have shown that all connecting trajectories u belong to a compact set $\bar{V}$.

Since we also have $b - a < \lambda_0$, bubbling-off is precluded by an energy estimate similar to (5.3). (Just a slightly more elaborate argument utilizing Gromov's compactness theorem shows that bubbling-off cannot occur whenever $b - a < h_2$ for some $h_2 > 0$, even if P is not spherically rational.) As a consequence, we infer by what have now become standard arguments (see, e.g., [HS, Sa]) that $\widehat{\mathcal{M}}(x, y)$ is compact when $\mu_{\mathrm{CZ}}(x) = \mu_{\mathrm{CZ}}(y) + 1$, and hence finite, and $\partial^2 = 0$. $\square$

From now on, when working with the Floer homology $\mathrm{HF}_*^{(a,\,b)}(H)$, we will always assume that $0 < a < b < \lambda_0$ and $b - a < h$ and that a and b are outside the action spectrum $\mathcal{S}(H)$ of H. (Note that, since P is spherically rational and H is compactly supported, $\mathcal{S}(H)$ is closed and has zero measure; see, e.g., [HZ3, Sc].)

5.2. **Properties.** The Floer homology spaces defined above have the standard properties of filtered Floer homology of compactly supported Hamiltonians on geometrically bounded manifolds; see [CGK, GG2, Gi8, Gü]. Here we recall only three of these properties that are explicitly used in the proof of the main theorem:

- For any three points $a < b < c$, where $0 < a < c < \lambda_0$ and $c - a < h$, we have the long exact sequence
$$\ldots \to \mathrm{HF}_*^{(a,\,b)}(H) \to \mathrm{HF}_*^{(a,\,c)}(H) \to \mathrm{HF}_*^{(b,\,c)}(H) \to \ldots .$$
- A monotone decreasing homotopy from a Hamiltonian H^+ to a Hamiltonian H^- gives rise to homomorphisms
$$\Psi_{H^+H^-} \colon \mathrm{HF}_*^{(a,\,b)}(H^+) \to \mathrm{HF}_*^{(a,\,b)}(H^-),$$
which are independent of the homotopy and commute with the long exact sequence homomorphisms.
- Let H^s be a family of Hamiltonians continuously parametrized by $s \in [0,\,1]$ and let $a(s) < b(s)$ be two continuous functions of s such that $a(s) < b(s)$ and all intervals $(a(s),\, b(s))$ satisfy the above requirements. Assume also that $a(s)$ and $b(s)$ are outside $\mathcal{S}(H^s)$. Then the groups $\mathrm{HF}_*^{(a(s),\,b(s))}(H^s)$ are isomorphic for all s. As a consequence, by continuity, we have $\mathrm{HF}_*^{(a,\,b)}(H)$ defined even when the orbits of H are degenerate.

The proofs of these properties and the constructions involved are identical to those for geometrically bounded symplectically aspherical manifolds (see [CGK, Gi8, GG2, Ke3]) which in turn follow closely the proofs for closed or convex manifolds (see, e.g., [BPS, FH, FS, Gi9, McSa, Sa, Sc, Vi] and references therein).

Remark 5.3. It is worth pointing that, in contrast with the total Floer–Novikov homology on a closed manifold, the filtered homology spaces $\mathrm{HF}_*^{(a,\,b)}(H)$ are not modules over the Novikov ring Λ of P. (See, e.g., [FHS, Sa, McSa] for the definition of Λ.) However, a part of the Λ-module structure is retained by the filtered homology. Namely, note first that the requirement that $0 < a < b < \lambda_0$ can be replaced by a less restrictive condition that $(a,\,b)$ contains no points of $\lambda_0\mathbb{Z}$. Then attaching a sphere $w \in \pi_2(P)$ simultaneously to all cappings gives rise to an isomorphism $\mathrm{CF}_*^{(a,\,b)}(H) \to \mathrm{CF}_{*+\mu}^{(a+\alpha,\,b+\alpha)}(H)$ of Floer complexes, and hence of Floer homology, where $\alpha = \langle \omega, w\rangle$ and $\mu = 2\,\langle c_1(TP), w\rangle$. This is an analogue of the action of the generators of Λ in the total Floer complex (or homology) of H.

Remark 5.4. If P is closed or geometrically bounded, some of the restrictions made in the construction of the filtered Floer homology can be relaxed. Namely, when P is closed and $W = P$, no homological conditions on P or restrictions on the action interval $(a,\,b)$ are needed, provided that $(a,\,b)$ is sufficiently short. (This readily follows from Gromov's compactness theorem, which guarantees that no bubbling-off can occur on Floer connecting trajectories with small energy.) When P is open and geometrically bounded, there is no need to fix an open set W with compact closure and the constant h is independent of the support of H. (This is a consequence of Sikorav's version of the Gromov compactness theorem; see [AL].) However, the assumptions that P is spherically rational and that $0 < a < b < \lambda_0$, or at least that $(a,\,b)$ contains no points of $\lambda_0\mathbb{Z}$, appear to be essential. In fact, it is not clear how to define the filtered Floer homology of a compactly supported Hamiltonian on a geometrically bounded open manifold without requiring P to be spherically rational.

6. Floer homological counterpart in the general case

Throughout this section, we assume that P satisfies the following two conditions:

(a) P is spherically rational and
(b) for any Hamiltonian on P, changing (the equivalence class of) a capping of a periodic orbit necessarily alters its action value, i.e., $\ker[\omega]|_{\pi_2(P)} \subset \ker c_1(TP)|_{\pi_2(P)}$.

Note that in the setting of Theorem 1.1 one can ensure that these requirements are met, as a consequence of either (i) or (ii), by replacing P by a small neighborhood of M.

6.1. **Generalization of Proposition 4.1.** Fix a symplectic tubular neighborhood W of M and an almost complex structure J on P. Let $K\colon P \to \mathbb{R}$ be an autonomous Hamiltonian attaining its Morse-Bott non-degenerate minimum $K = 0$ along a closed symplectic submanifold $M \subset P$ and let H be defined exactly as in Section 4.2 with $\{K \leq r^2 + \epsilon_0\} \subset W$. By (a), in the notation of Section 5, the filtered Floer homology $\mathrm{HF}_*^{(a,\,b)}(H)$ is defined whenever $0 < a < b < \lambda_0$ and $b - a < h = h(W, J)$. Then, the following analogue of Proposition 4.1 holds:

Proposition 6.1. *There exists a function $C(r) > 0$ of $r > 0$ such that $C(r) \to 0$ as $r \to 0$ and, once $r > 0$ is sufficiently small, we have*

$$\mathrm{HF}_{n_0}^{(a,\,b)}(H) \neq 0$$

for any H as above with $\max H = C(r)$ and for some interval $(a,\,b)$ with $C(r) < a < b < 2C(r)$. (Here, as in Proposition 4.1, $n_0 = 1 + (\operatorname{codim} M - \dim M)/2$.)

Remark 6.2. Since the proposition concerns only a small neighborhood of M, requirement (a) can be replaced by the condition that M is spherically rational and, in (b), $[\omega]|_{\pi_2(P)}$ can be replaced $[\omega]|_{\pi_2(M)}$.

6.2. **Proof of Proposition 6.1.** To prove the proposition we will, as in [GG2], construct functions $F^{\pm}$ such that

$$F^- \leq H \leq F^+$$

and $\mathrm{HF}^{(a,\,b)}_{n_0}(F^{\pm}) \cong \mathbb{Z}_2$ and the monotone homotopy map

$$\Psi\colon \mathbb{Z}_2 \cong \mathrm{HF}^{(a,\,b)}_{n_0}(F^+) \to \mathrm{HF}^{(a,\,b)}_{n_0}(F^-) \cong \mathbb{Z}_2$$

is an isomorphism if $r > 0$ is sufficiently small. Then $\mathrm{HF}^{(a,\,b)}_{n_0}(H) \neq 0$, for Ψ factors through $\mathrm{HF}^{(a,\,b)}_{n_0}(H)$. The argument closely follows, with some simplifications, the proof of Proposition 4.1 given in [GG2] and we only briefly outline its key elements.

6.2.1. *Functions $F^{\pm}$ and the parameters a, b and $C(r)$.* The almost complex structure J and the symplectic form ω give rise to a Hermitian metric on the normal bundle $(TM)^{\omega}$ to M. Without loss of generality we may assume that W is a tubular neighborhood of M in P; see Section 3.2.3. In particular, W is equipped with projection $W \to M$ whose fibers are identified with fiber-wise balls in the normal bundle $(TM)^{\omega}$ and this identification preserves the symplectic structure. Denote by $\rho\colon W \to \mathbb{R}$ the square of the Hermitian norm on $(TM)^{\omega}$ divided by 4π, i.e., $\rho(X) = \|X\|^2/(4\pi)$ when X is viewed as a point in $(TM)^{\omega}$. It is easy to see that all levels of ρ are comprised of one-periodic orbits of its Hamiltonian flow. These orbits are the Hopf circles lying in the fibers and bounding symplectic area $4\pi^2\rho$; see, e.g., [CGK, GG2].

The normal-direction Hessian $d^2_M K$ along M can also be viewed as a fiber-wise quadratic function $d^2_M K\colon W \to \mathbb{R}$. (Since K is Morse–Bott non-degenerate, $d^2_M K$ is a fiber-wise metric on $(TM)^{\omega}$. In general, this metric is not Hermitian.) Recall also that $0 < \epsilon < \epsilon_0 = r^2/10$; see Section 4.2. Thus, when $r > 0$ is small, the shell $r^2 - \epsilon \leq K \leq r^2 + \epsilon$ is contained in the shell

$$Z = \{r^2 - 2\epsilon_0 \leq d^2_M K \leq r^2 + 2\epsilon_0\}.$$

Set

$$\rho_3^- = \min_Z \rho, \quad \rho_2^- = 2\rho_3^-/3 \quad \text{and} \quad \rho_1^- = \rho_3^-/3,$$

so that the points ρ_1^- and ρ_2^- divide the interval $[0,\, \rho_3^-]$ into three equal parts. Furthermore, let, as in Fig. 1,

$$\rho_1^+ = \max_Z \rho, \quad \rho_2^+ = \rho_1^+ + \rho_1^- \quad \text{and} \quad \rho_3^+ = \rho_1^+ + 2\rho_1^-,$$

and

$$C(r) = 8\pi^2\rho_3^+.$$

In other words, $C(r)/2$ is the symplectic area bounded by one-periodic orbits of ρ on the level $\rho = \rho_3^+$. It is clear that $C(r) \to 0$ as $r \to 0$. From now on, we assume that $\max H = C(r)$.

The function $F^-\colon [0,\, \rho_3^-] \to \mathbb{R}$ is defined as follows (see Fig. 1):

- $F^-(\rho) \equiv C(r)$ for $\rho \in [0, \rho_1^-]$;
- $F^-(\rho) \equiv 0$ for $\rho \in [\rho_2^-, \rho_3^-]$;

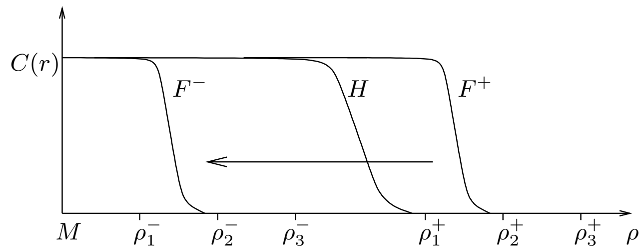

FIGURE 1. The functions $F^\pm$ and the homotopy.

- $F^-(\rho)$ is a linear function on $[\rho_1^-, \rho_2^-]$ ranging from $C(r)$ to 0 with irrational slope, except for ρ close to ρ_1^- and ρ_2^- where F^- is concave (with decreasing $(F^-)' \leq 0$) and, respectively, convex (with increasing $(F^-)' \leq 0$).

We extend the function $F^- \circ \rho$ from the domain $\{\rho < \rho_3^-\}$ to P by setting it to be identically zero outside the domain and, abusing notation, refer to the resulting Hamiltonian on P as F^-.

It is essential that the ratio $C(r)/\rho_1^-$ is independent of r (since $d_M^2 K$ and ρ are both fiber-wise quadratic), and hence the slope of F^- can also be taken independent of r.

Let also $F^+ \colon [0, \rho_3^+] \to \mathbb{R}$ be defined by

- $F^+(\rho) \equiv C(r)$ for $\rho \in [0, \rho_1^+]$;
- $F^+(\rho) = F^-(\rho - \rho_1^+ + \rho_1^-)$ for $\rho \in [\rho_1^+, \rho_2^+]$;
- $F^+(\rho) \equiv 0$ for $\rho \in [\rho_2^+, \rho_3^+]$.

In other words, F^+ is obtained from F^- by shifting the graph of F^- to the left by $\rho_1^+ - \rho_1^-$ and extending it to the remaining interval $[0, \rho_1^+ - \rho_1^-]$ as a function identically equal to $C(r)$. In particular, F^+ has the same slope as F^- and this slope is independent of r. Finally, we extend $F^+ \circ \rho$ to P in the same fashion as $F^- \circ \rho$ and again keep the notation F^+ for the resulting Hamiltonian.

By the construction, $F^- \leq H \leq F^+$. Set

$$a = C(r) + 2\pi^2 \rho_1^- \quad \text{and} \quad b = C(r) + 6\pi^2 \rho_3^+.$$

Then $a > C(r)$ and, since $C(r) = 8\pi^2 \rho_3^+$, we have $b < 2C(r)$. In what follows, we will assume that $r > 0$ is sufficiently small. Thus, in particular, $2C(r) < \lambda_0$ and $b - a < C(r) < h(W, J)$, and the Floer homology groups in question are defined.

6.2.2. *One-periodic orbits of $F^\pm$.* Trivial periodic orbits of $F^\pm$ are either the points where $F^\pm = C(r)$ or the points where $F^\pm = 0$. Non-trivial one-periodic orbits of the functions $F^\pm$ fill in entire energy levels of $F^\pm$. We break down the corresponding energy values into two groups $\rho = x_l^\pm$ and $\rho = y_l^\pm$ for each of the functions $F^\pm$; see Fig. 2.

The first group of levels is located in the region where $F^\pm$ is convex. We label these levels by the corresponding values of ρ in the increasing order:

$$x_1^\pm < x_2^\pm < \ldots < x_k^\pm,$$

where all $x_j^\pm$ are close to and slightly greater than $\rho_1^\pm$.

The levels from the second group are located in the region where $F^\pm$ is concave. Again, we label these levels by the corresponding values of ρ, but now in the

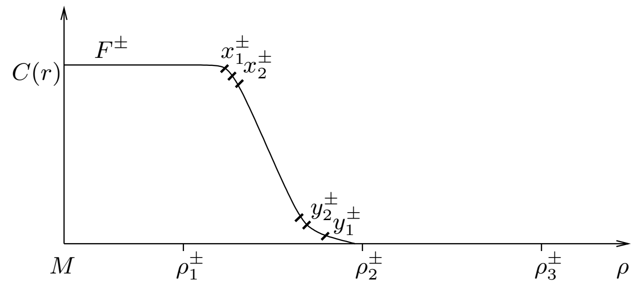

FIGURE 2. The energy levels $x_l^{\pm}$ and $y_l^{\pm}$.

decreasing order:

$$y_k^{\pm} < \ldots < y_2^{\pm} < y_1^{\pm},$$

where all $y_l^{\pm}$ are close to and slightly smaller than $\rho_2^{\pm}$.

Note that the number k of levels in every group is completely determined by the slope of the Hamiltonian. In particular, k is the same for all groups and is independent of r.

One-periodic orbits of $F^{\pm}$ on the levels $\rho = x_l^{\pm}$ and $\rho = y_l^{\pm}$ are the fiber-wise Hopf circles traversed l-times. We equip these orbits with cappings by discs contained in the fibers and refer to this capping as fiber-wise. Denote by $A(x_l^{\pm})$ and $A(y_l^{\pm})$ the resulting action values. (The action is independent of the choice of an orbit on the level.) It is easy to see that

$$A(x_l^{\pm}) = C(r) + 4\pi^2 l x_l^{\pm} + \ldots = C(r) + 4\pi^2 l \rho_1^{\pm} + \ldots$$

and

$$A(y_l^{\pm}) = 4\pi^2 l y_l^{\pm} + \ldots = 4\pi^2 l \rho_2^{\pm} + \ldots,$$

where the dots denote an error that can be made arbitrarily small by making $F^{\pm}$ close to a piece-wise linear function; see [GG2]. Note that $A(x_1^{\pm})$ is in $(a,\, b)$ while $A(y_1^{\pm})$ and $A(y_2^{\pm})$ are outside this interval.

We require $r > 0$ to be so small that all $A(x_l^{\pm})$ and $A(y_l^{\pm})$ are in the range from 0 to $h(W, J)$. This condition is indeed met for small $r > 0$, as k is independent of r and the largest of these actions does not exceed $\max\{A(x_k^{+}), A(y_k^{+})\} \leq C(r) + 4\pi^2 k \rho_2^{\pm} + \ldots$.

6.2.3. *Floer homology of $F^{\pm}$ and the monotone homotopy map Ψ.* Let $(\alpha,\, \beta)$ be an interval in $(a,\, b)$ containing only one point of $\mathcal{S}(F^{\pm})$. Denote by Σ the unit sphere bundle in $(TM)^{\omega}$. (Thus, the levels $\rho = x_l^{\pm}$ and $\rho = y_l^{\pm}$ are diffeomorphic to Σ.)

Lemma 6.3. $\mathrm{HF}_*^{(\alpha,\, \beta)}(F^{\pm}) = H_{*-\kappa}(\Sigma; \mathbb{Z}_2)$, *where the shift of degrees κ depends on the level.*

We defer the proof of the lemma to Section 6.3 and continue the proof of the proposition.

To determine the exact value of the shift κ, consider a non-degenerate time-dependent perturbation $\tilde{F}^{\pm}$ of $F^{\pm}$ that differs from $F^{\pm}$ only in small neighborhoods of the levels $x_l^{\pm}$ and $y_l^{\pm}$. The perturbation $\tilde{F}^{\pm}$ can be explicitly constructed (see

[GG2, Section 5.2.5]) so that every level $x_l^{\pm}$ and $y_l^{\pm}$ splits into a number of non-degenerate orbits contained in the fibers of W and these orbits, equipped with fiber-wise cappings, have Conley–Zehnder indices in the intervals

$$\begin{aligned}&[(2l-1)q-m+1,\,(2l+1)q+m] \quad \text{for the level } \rho=x_l^{\pm} \text{ and}\\ &[(2l-1)q-m,\,(2l+1)q+m-1] \quad \text{for the level } \rho=y_l^{\pm},\end{aligned}$$

where $2m=\dim M$ and $2q=\operatorname{codim} M$. Note that the length of each of these intervals is $2(m+q)-1=\dim\Sigma$. Hence, κ is the left end-point of the index interval, i.e.,

$$\kappa=(2l-1)q-m+1 \text{ for } \rho=x_l^{\pm} \text{ and } \kappa=(2l-1)q-m \text{ for } \rho=y_l^{\pm}.$$

In particular, $\kappa=n_0$ for $x_1^{\pm}$ and, by Lemma 6.3, $\mathrm{HF}_{n_0}^{(\alpha_1,\,\beta_1)}(F^{\pm})=\mathbb{Z}_2$ when $(\alpha_1,\,\beta_1)$ is a small interval containing no other points of $\mathcal{S}(F^{\pm})$ than $A(x_1^{\pm})$.

Arguing as in [GG2], we establish the equality $\mathrm{HF}_{n_0}^{(a,\,b)}(F^{\pm})=\mathbb{Z}_2$ by utilizing the Floer homology long exact sequence; see Section 5.

First note that the only action values of $F^{\pm}$ and $\tilde{F}^{\pm}$ in $(a,\,b)$ are those of the orbits from some of the levels $x_l^{\pm}$ and $y_l^{\pm}$, equipped with fiber-wise cappings. In other words, only the fiber-wise cappings are relevant. This is a consequence of (b) and the requirement that $r>0$ is so small that all $A(x_l^{\pm})$ and $A(y_l^{\pm})$ are in $(0,\lambda_0)$. Consider now a family of intervals with the left end-point sliding down from α_1 to a and the right end-point increasing from β_1 to b. The filtered Floer homology of $F^{\pm}$ can change only when an end-point of the interval moves through an action value. More specifically, the homology in degree n_0 can be effected only when an end-point moves through an action value $A(x_l^{\pm})$ or $A(y_l^{\pm})$ with index interval containing n_0-1 or n_0 or n_0+1. The only action values with index intervals containing n_0-1 or n_0 are $A(y_1^{\pm})$ and $A(y_2^{\pm})$, which are, however, outside the interval $(a,\,b)$ since $A(y_1^{\pm})<A(y_2^{\pm})<a$. The levels $x_l^{\pm}$ with $l\geq 2$ and $y_l^{\pm}$ with $l\geq 3$ have index intervals starting above n_0+1. Finally, $A(x_1^{\pm})\in(\alpha_1,\,\beta_1)\subset(a,\,b)$. Hence, as readily follows from the long exact sequence, $\mathrm{HF}_{n_0}^{(a,\,b)}(F^{\pm})\cong\mathrm{HF}_{n_0}^{(\alpha_1,\,\beta_1)}(F^{\pm})\cong\mathbb{Z}_2$.

The fact that the monotone homotopy map Ψ is an isomorphism is established in a similar fashion. Consider a monotone homotopy F^s from F^+ to F^- indicated by the arrow in Fig. 1 and obtained by sliding the graph of F^+ to the left until it matches the graph of F^-. By the long exact sequence, $\mathrm{HF}_{n_0}^{(a,\,b)}(F^s)$ can change only when action values of F^s with index interval containing n_0-1 or n_0 or n_0+1 enter or leave the interval $(a,\,b)$. This, however, never happens, as is clear from the calculation of actions and index intervals for $F^{\pm}$; see [GG2] for more details.

To complete the argument, it remains to prove Lemma 6.3.

6.3. **Local Floer homology and the proof of Lemma 6.3.** When P is geometrically bounded and symplectically aspherical, the lemma is an immediate consequence of the results of Poźniak, [Poz], and, in particular, of [Poz, Corollary 3.5.4]; see also [GG2] and [BPS]. Moreover, Corollary 3.5.4 from [Poz] can be extended to a broader class of manifolds to apply in the setting of Lemma 6.3. However, such an extension is not entirely straightforward even though it is essentially a consequence of Poźniak's calculation of local Lagrangian Floer homology for clean intersections, and we prefer to give a simpler *ad hoc* proof.

6.3.1. *Local Floer homology.* In this section we briefly recall the construction of local Floer homology for autonomous Hamiltonians, following [Fl5]; see also, e.g., [BPS, Gi9, Poz].

Consider an autonomous Hamiltonian F on a spherically rational symplectic manifold P^{2n} or, more generally, on an open subset of P invariant under the flow of F. Let Σ be a compact, connected set of fixed points of $\varphi = \varphi_F^1$. Then Σ is automatically invariant under the flow of F and hence comprised entirely of one-periodic orbits of F. In what follows, we will also assume that Σ is isolated, i.e., every fixed point of φ in some neighborhood of Σ (an isolating neighborhood) is necessarily a point of Σ.

Fix an isolating neighborhood U of Σ and let $\tilde{F}$ be a one-periodic in time perturbation of F such that $F - \tilde{F}$ is C^2-small and supported in U, and all one-periodic orbits of $\tilde{F}$ in U are non-degenerate. Let $\mathrm{CF}_k(\tilde{F}, U)$ be the vector space over $\mathbb{Z}_2$ generated by capped one-periodic orbits of $\tilde{F}$ in U of index k. (Note that we do not require the cappings to be contained in U.) Furthermore, fix $\epsilon > 0$ and a one-periodic almost complex structure J on P, and define the Floer differential $\partial_\epsilon \colon \mathrm{CF}_k(\tilde{F}, U) \to \mathrm{CF}_{k-1}(\tilde{F}, U)$ by the standard formula (5.1), where now all Floer connecting trajectories are required to have energy smaller than ϵ.

The standard argument shows that $\partial_\epsilon^2 = 0$ whenever $\epsilon > 0$ and $\|F - \tilde{F}\|_{C^2}$ are small enough, and, moreover, the resulting local Floer homology spaces $\mathrm{HF}_*(F, \Sigma)$ are independent of $\epsilon > 0$, $\tilde{F}$ and J; see [Fl5] and also, e.g., [BPS, Gi9, HS, McSa, Poz, Sa, SZ]. (To be more precise, here we need to require that $\epsilon < \epsilon_0(U, J)$ and $\|F - \tilde{F}\|_{C^2} < \delta_0(U, J, \epsilon)$. Then the Floer anti-gradient trajectories connecting periodic orbits in U are confined to U due to the energy estimates from, e.g., [SZ, Sa]. Bubbling-off cannot occur, for $\epsilon < \lambda_0$. As a consequence, the standard compactness and continuation arguments apply.)

Moreover, the complex $\mathrm{CF}_*(\tilde{F}, U)$ carries a natural action filtration (see Section 5) and we denote the resulting filtered local Floer homology by $\mathrm{HF}_*^{(a,\, b)}(F, \Sigma)$. This homology spaces are well-defined only when the points a and b are outside the action spectrum $\mathcal{S}(F, \Sigma)$ of $F|_\Sigma$. (By definition, $\mathcal{S}(F, \Sigma)$ is comprised of action values of one-periodic orbits of F in Σ with all possible cappings. It is easy to see that $\mathcal{S}(F, \Sigma)$ has zero measure and is closed and nowhere dense since P is spherically rational.) The filtered Floer homology is essentially "localized" at the points of $\mathcal{S}(F, \Sigma)$ and hence here, in contrast with the global case, the filtration plays a rather superficial role. We will use it only to distinguish contributions from different cappings of the same orbit.

Filtered local Floer homology inherits, in an obvious way, most of the properties of ordinary filtered Floer homology. We will need the following standard invariance result (cf. [BPS, Poz, Gi9, Vi]):

- Let F^s be a family of Hamiltonians continuously parametrized by $s \in [0,\, 1]$ and such that Σ is an isolated set of periodic orbits for all F^s. Let $a(s) < b(s)$ be two continuous functions of s such that $a(s)$ and $b(s)$ are outside $\mathcal{S}(F^s, \Sigma)$. Then the groups $\mathrm{HF}_*^{(a(s),\, b(s))}(F^s, \Sigma)$ are isomorphic for all s.

Let us now turn to some examples which are relevant to the proof.

Example 6.4. Assume that P is symplectically aspherical and hence $\mathcal{S}(F, \Sigma)$ is comprised of one point $c \in \mathbb{R}$. Then $\mathrm{HF}_*^{(a,\, b)}(F, \Sigma) = 0$ when c is outside $[a,\, b]$ and

$\mathrm{HF}_*^{(a,\,b)}(F,\Sigma) = \mathrm{HF}_*(F,\Sigma)$ if $a < c < b$. Assume furthermore that Σ is a Morse-Bott non-degenerate manifold of fixed points of φ, i.e., $\ker(d\varphi_p - I) = T_p\Sigma$ for all $p \in \Sigma$. Then $\mathrm{HF}_*(F,\Sigma) = H_{*-\kappa}(\Sigma;\mathbb{Z}_2)$ as is proved in [Poz].

Example 6.5. In this example, we assume that P^{2n} satisfies condition (b) in addition to being spherically rational (condition (a)). Let Σ be a Morse-Bott non-degenerate critical manifold of a smooth function F with, say, $F|_\Sigma = 0$. Then Σ is an isolated set of fixed points of φ, and

$$\mathrm{HF}_*^{(a,\,b)}(F,\Sigma) = H_{*-\kappa(F)}(\Sigma;\mathbb{Z}_2), \tag{6.1}$$

whenever Σ is a hypersurface (or, more generally, a coisotropic submanifold) and $(a,\,b)$ is a short interval containing 0. Here $\kappa(F) = n - \mathrm{index}(\Sigma)$, where $\mathrm{index}(\Sigma)$ is the index of F at Σ. Note that Σ is also a Morse-Bott non-degenerate manifold of fixed points of φ since Σ is coisotropic. Thus, if P is symplectically aspherical, the identification (6.1) becomes a particular case of Poźniak's result mentioned in the previous example.

To establish the general case of (6.1), we argue as follows. First note that $\mathrm{HF}_*^{(a,\,b)}(sF,\Sigma)$ does not change as s ranges from 1 to some small value $s_0 > 0$ such that s_0F is C^2-small. (Here, again we use the assumption that Σ is coisotropic which guarantees that Σ is an isolated fixed point set for all $s \in (0,\,1]$. Furthermore, it is clear from (a) and (b) that the end points a and b are not in $\mathcal{S}(sF)$, provided that a and b are sufficiently close to zero.) Then, $\mathrm{HF}_*^{(a,\,b)}(s_0F,\Sigma)$ can be identified, up to a shift of degree by n, with the local Morse homology of F at Σ by arguing as in [FHS, HS, SZ] and using again conditions (a) and (b); cf. [Gi9]. (Alternatively, one can utilize the PSS isomorphism, [PSS], not relying on (b).) Finally, it is a standard fact that the local Morse homology in question is equal to $H_*(\Sigma;\mathbb{Z}_2)$, up to a shift of degree by $\mathrm{index}(\Sigma)$.

Under suitable additional hypotheses, the filtered local Floer homology of F is equal to the filtered global Floer homology. Namely, let F and P and $(a,\,b)$ be as in Section 5. Assume that $(a,\,b)$ contains only one point of $\mathcal{S}(F)$ and that this point also belongs to $\mathcal{S}(F,\Sigma)$. Then, as is easy to see,

$$\mathrm{HF}_*^{(a,\,b)}(F,\Sigma) = \mathrm{HF}_*^{(a,\,b)}(F). \tag{6.2}$$

Remark 6.6. The construction of local Floer homology outlined above goes through with obvious modifications even when F is not autonomous; see [Fl5] and, e.g., [BPS, Poz]. The only reason that F is assumed here to be independent of time is that this assumption makes the construction much more explicit, simplifies the wording, and is sufficient for the proof of Lemma 6.3.

6.3.2. *Proof of Lemma 6.3.* Let F be one of the two functions $F^\pm$ and let $\Sigma = \{\rho = z\}$, where $z = x_l^\pm$ or $z = y_l^\pm$, be one of the levels in question. Consider the Hamiltonians $F_0(\rho) = F(z) + F'(z)(\rho - z)$ and $f = F - F_0$ defined on a neighborhood of Σ. Then the hypersurface Σ is a Morse-Bott non-degenerate critical manifold of f and $f|_\Sigma = 0$. By Example 6.5, $\mathrm{HF}_*^{(\alpha',\,\beta')}(f,\Sigma) = H_{*-\kappa(f)}(\Sigma;\mathbb{Z}_2)$, when $(\alpha',\,\beta')$ is a short interval containing 0.

Denote by c the action of F on Σ with respect to the fiber-wise capping, i.e., $c = A(z)$, and set $\alpha = \alpha' + c$ and $\beta = \beta' + c$. We will show that

$$\mathrm{HF}_*^{(\alpha,\,\beta)}(F,\Sigma) = \mathrm{HF}_{*-\kappa'}^{(\alpha',\,\beta')}(f,\Sigma) \tag{6.3}$$

for some shift of degrees κ'. This will prove the lemma, since then, due to (6.1) and (6.2),

$$\mathrm{HF}_*^{(\alpha,\,\beta)}(F) = \mathrm{HF}_*^{(\alpha,\,\beta)}(F,\Sigma) = H_{*-\kappa}(\Sigma;\mathbb{Z}_2),$$

where $\kappa = \kappa' + \kappa(f)$. (Note that c and κ' can be interpreted as the action and, respectively, the Maslov index of the loop ψ^t, cf. [Gi9, Sections 2.3 and 3.2].)

To establish (6.3), consider a perturbation $\tilde{f}$ of f in a small neighborhood U of Σ as in the construction of local Floer homology. Without loss of generality we may assume that $\tilde{f}$ is autonomous and all one-periodic orbits of $\tilde{f}$ in U are critical points of $\tilde{f}$ and that all such critical points are located on Σ. Then these orbits enter $\mathrm{CF}_*^{(\alpha',\,\beta')}(\tilde{f},U)$ equipped with trivial cappings. For any other capping would necessarily, by (b), move the action outside the range $(\alpha',\,\beta')$. Let $J = J_t$ be a (time-dependent) almost complex structure. A Floer anti-gradient trajectory u connecting two one-periodic orbits of $\tilde{f}$ in U is a sphere. (We are assuming that $\tilde{f}$ is so close to f that u is contained in U.) Since, by the definition of local Floer homology, the energy of u is small and the values of $\tilde{f}$ at its critical points are close to zero, the symplectic area of u is small. Therefore, by (a) and (b),

$$\langle \omega, u\rangle = 0 = \langle c_1(TP), u\rangle\,. \tag{6.4}$$

The Hamiltonian flow of F_0 is a one-periodic loop $\psi^t = \varphi^t_{F_0}$ of fiber-wise rotations and the Hamiltonian $\tilde{F} = F_0 + \tilde{f}\circ(\psi^t)^{-1}$ generating the flow $\psi^t\circ\varphi^t_{\tilde{f}}$ is a small perturbation of F. Denote by $\mathrm{HF}_*^{(\alpha',\,\beta')}(\tilde{f},U)$ and $\mathrm{HF}_*^{(\alpha,\,\beta)}(\tilde{F},U)$ the homology of the complexes of $\mathrm{CF}_*^{(\alpha',\,\beta')}(\tilde{f},U)$ and, respectively, $\mathrm{CF}_*^{(\alpha,\,\beta)}(\tilde{F},U)$. By the definition of local Floer homology,

$$\mathrm{HF}_*^{(\alpha',\,\beta')}(\tilde{f},U) = \mathrm{HF}_*^{(\alpha',\,\beta')}(f,\Sigma) \quad\text{and}\quad \mathrm{HF}_*^{(\alpha,\,\beta)}(\tilde{F},U) = \mathrm{HF}_*^{(\alpha,\,\beta)}(F,\Sigma),$$

and (6.3) is equivalent to the isomorphism

$$\mathrm{HF}_*^{(\alpha,\,\beta)}(\tilde{F},U) \cong \mathrm{HF}_{*-\kappa'}^{(\alpha',\,\beta')}(\tilde{f},U). \tag{6.5}$$

This isomorphism is induced by the composition with the loop ψ^t.

Indeed, the composition with ψ^t gives rise to a one-to-one correspondence between one-periodic orbits of $\tilde{f}$ contained in U and those of $\tilde{F}$, and the latter are fiber-wise Hopf circles (perhaps, multi-covered). Furthermore, as is well known, $\psi(u)(s,t) := \psi^t(u(s,t))$ is a Floer anti-gradient trajectory for $\tilde{F}$ in U with respect to the almost complex structure $\psi(J) := d\psi^t\circ J_t\circ(d\psi^t)^{-1}$ if and only if u is a Floer anti-gradient trajectory for $\tilde{f}$ and J contained in U; see, e.g., [Gi9, Sc]. (Moreover, the regularity requirements are satisfied for $(\tilde{f},J)$ if and only if they are satisfied for $(\tilde{F},\psi(J))$.)

Let us equip the one-periodic orbits of $\tilde{F}$ contained in U with fiber-wise cappings. Then these capped orbits are the only orbits entering $\mathrm{CF}_*^{(\alpha,\,\beta)}(\tilde{F},U)$, as again follows from (a) and (b). Hence, to establish (6.5) and thus finish the proof of the lemma, it is sufficient to show that the Floer connecting trajectories $\psi(u)$ are compatible with such cappings. In other words, it remains to prove that whenever $\psi(u)$ is a Floer anti-gradient trajectory from γ_0 and γ_1 and v_0 and v_1 are cappings of γ_0 and γ_1 by fiber-wise Hopf discs (perhaps, multi-covered), the capping v_1 is equivalent to the capping $v_0\#\psi(u)$ obtained by attaching v_0 to $\psi(u)$. To this end, consider the sphere w obtained by attaching the suitably oriented discs v_0 and v_1

to $\psi(u)$. The cappings $v_0 \# \psi(u)$ and v_1 are equivalent if and only if

$$\langle \omega, w \rangle = 0 = \langle c_1(TP), w \rangle .$$

Let $\pi\colon W \to M$ be the tubular neighborhood projection. Since v_0 and v_1 lie in the fibers of π, the projections $\pi(v_0)$ and $\pi(v_1)$ are points. Hence, $\pi(w)$ is homotopic to $l \cdot \pi(u)$, where $l \in \mathbb{Z}$, for ψ^t is comprised of fiber-wise rotations. Then, by (6.4),

$$\langle c_1(TP), w \rangle = \langle c_1(TP|_M), \pi(w) \rangle = l \, \langle c_1(TP|_M), \pi(u) \rangle = l \, \langle c_1(TP), u \rangle = 0.$$

A similar calculation, showing that $\langle \omega, w \rangle = 0$, completes the proof of (6.5) and of the lemma.

7. Proof of the main theorem

7.1. **Proof of Theorem 1.1.** When (i) holds, the proof of the main theorem in the general case is identical word-for-word to the proof for geometrically bounded, symplectically aspherical manifolds (Section 4) with Proposition 6.1 used in place of Proposition 4.1.

To establish the theorem when (ii) holds, we use, in addition to the Sturm comparison theorem for K, the action bounds from Proposition 6.1 to control the effect of capping on the Salamon–Zehnder invariant.

Fix small parameters $r > 0$ and $\epsilon > 0$ with $\epsilon < r^2/10$ and consider a Hamiltonian H as in Section 6.1. Recall that (ii) implies that the hypotheses (a) and (b) of Section 6.1 are satisfied. Then, as in Section 4, it readily follows from Proposition 6.1 that H has a (non-trivial) one-periodic orbit γ with capping v such that

$$1 - \dim M = n_0 - \dim P/2 \leq \Delta_v(\gamma, H) \leq n_0 + \dim P/2 = 1 + \operatorname{codim} M \qquad (7.1)$$

and

$$a \leq A_H(\gamma, v) \leq b.$$

Recall that $0 < C(r) < a < b < 2C(r)$ and $0 \leq H \leq C(r)$, where $C(r) \to 0$ as $r \to 0$. Hence, the symplectic area of v is *a priori* bounded:

$$\left| \int_v \omega \right| \leq const, \qquad (7.2)$$

where $const$ is independent of r and ϵ and the Hamiltonian H. (Throughout the rest of the proof we adopt the notational convention from Section 3: in all expressions $const$ will stand for a constant which is independent of r, ϵ, H, and (γ, v), provided that r is sufficiently small. The value of this constant is allowed to vary from one formula to another. A similar convention is also applied to the constants $a > 0$ and b and c.)

As in Section 4, we may view γ as a T-periodic orbit of K since H is a function of K and the orbit γ is non-trivial. Then, by (3.6),

$$l(\gamma) \leq const \cdot r \cdot T. \qquad (7.3)$$

Fix a 2-form σ representing $c_1(TP)$. By (ii), $\sigma = \lambda\omega + d\alpha$, with $\lambda \neq 0$, for some 1-form α. Then

$$\left| \int_v \sigma \right| \leq \left| \lambda \int_v \omega \right| + \left| \int_\gamma \alpha \right|,$$

and, from (7.2) and (7.3), we see that

$$\left| \int_v \sigma \right| \leq const_1 \cdot r \cdot T + const_2. \qquad (7.4)$$

By Lemma 2.6, we have $\Delta_v(\gamma, K) = -\Delta_v(\gamma, H)$, where the negative sign is a consequence of the fact that H is a decreasing function of K. Thus, (7.1) turns into

$$1 - \dim M \leq -\Delta_v(\gamma, K) \leq 1 + \operatorname{codim} M,$$

and, by Proposition 3.2,

$$-\Delta_v(\gamma, K) \geq a \cdot T - c - 2 \int_v \sigma,$$

where $a > 0$. (The negative sign is a result of the convention change from Section 3 to Section 4.) Therefore,

$$T - \frac{2}{a} \int_v \sigma \leq \frac{c + 1 + \operatorname{codim} M}{a}.$$

Combining this upper bound with (7.4), we conclude that if $r > 0$ is sufficiently small, $T \leq T_0$ for some T_0 that depends only on K.

Finally, as in the proof of the particular case, passing to the limit as $\epsilon \to 0$, we see that the T-periodic orbits γ of K converge, by the Arzela-Ascoli theorem, to a periodic orbit of K on the level $K = r^2$ with period bounded from above by T_0. This completes the proof of Theorem 1.1.

Remark 7.1. As is readily seen from the proofs of Theorem 1.1 and Proposition 6.1, one can also estimate the action and the symplectic area of the orbit γ of K on the level $K = r^2$. Namely, let v be the capping of γ as in the proof of Theorem 1.1. Then,

$$-const_1 \cdot r^2 \leq \int_v \omega \leq -const_2 \cdot r^2 \quad \text{and} \quad |A_K(\gamma, v)| \leq const \cdot r^2,$$

where all constants are positive and depend only on K. This follows from the facts that $const_1 \cdot r^2 \leq C(r) < const_2 \cdot r^2$ with, perhaps, some other values of the constants and that the period of γ is bounded from above.

7.2. **Concluding remarks.**

7.2.1. *Dense existence of periodic orbits.* Proposition 6.1 can be reformulated (with some loss of information) as a dense existence theorem for periodic orbits of K:

Proposition 7.2. *Assume that P satisfies conditions (a) and (b) of Section 6.1. Then, for a dense set of small $r > 0$, the level $K = r^2$ carries a contractible in P periodic orbit γ with capping v such that*

$$-1 - \operatorname{codim} M \leq \Delta_v(\gamma) \leq \dim M - 1 \quad \textit{and} \quad 0 < \left| \int_v \omega \right| < const \cdot r^2,$$

where const depends only on K.

Referring the reader back to Section 1.3 for a discussion of other dense or almost existence results, here we only point out that almost existence of contractible periodic orbits of K without upper and lower bounds on $\Delta_v(\gamma)$ is proved in [Lu2] under no topological assumptions on P. Proposition 7.2 is sufficient for the proof of Theorem 1.1 and can be easily established as a consequence of Proposition 6.1 by passing from periodic orbits of H to those of K as in Sections 4 and 7.1.

7.2.2. *The role of hypotheses (i) and (ii).* As is mentioned in Section 1, hypotheses (i) and (ii) in Theorem 1.1 can possibly be relaxed. Indeed, the Sturm theoretic counterpart of the proof (Proposition 3.2) requires no topological assumptions on P or M. The Floer homological part of the argument (Proposition 6.1) holds under hypotheses (a) and (b), less restrictive than (i) or (ii), and can probably be extended to, at least, all spherically rational manifolds by using, for instance, the machinery of central Floer homology from [Ke4]; see also [Al]. Furthermore, in the form of Proposition 7.2, it can perhaps be generalized to arbitrary symplectic manifolds by utilizing the holomorphic curve techniques as in, e.g., [Lu2]. However, it is not clear to the authors how to combine these two counterparts to obtain an upper bound on the period without using conditions (i) or (ii) or some other condition relating $[\omega]$ and $c_1(TP)$.

7.2.3. *Action control and "contact homology" approach.* We conclude this paper by discussing two approaches to proving Theorem 1.1, which are more natural than the one used here but encounter a serious difficulty.

The key to the first approach lies in establishing an upper bound on the period of an orbit of H via its action. Then, the theorem would follow directly from a version of Proposition 6.1. This method has been used, for instance, to prove the Weinstein conjecture for hypersurfaces (of contact type) as a consequence of a calculation of Floer or symplectic homology; see, e.g., [FHW, HZ3] and [Gi7] for further references. Here the condition that the level in question has contact type is crucial for controlling the period of an orbit via its action. This can be seen, for instance, from the counterexamples to the Hamiltonian Seifert conjecture, [Gi4, Gi5, GG1, GG2, Ke2]. In the setting of Theorem 1.1, the energy levels $S = \{K = r^2\}$ do not in general have contact type (with very few exceptions), and the authors are not aware of any way to relate the period and the action in this case by merely using the fact that S is fiber-wise convex.

The idea of the second approach is to make use of a version of the contact homology $\mathrm{HC}_*(S)$ defined for the level S and detecting closed characteristics on S. (Strictly speaking, no construction of HC_* applicable to the levels in question is available at the moment.) Then, one would consider the continuation map $\mathrm{HC}_*(\Sigma^+) \to \mathrm{HC}_*(S) \to \mathrm{HC}_*(\Sigma^-)$, where Σ^+ is a level of ρ enclosing S and Σ^- is a level of ρ enclosed by S. As in the proof of Proposition 6.1, one can expect this map to be non-zero, which would then yield $\mathrm{HC}_*(S) \neq 0$. This argument relies on the assumption that the groups HC_* are sufficiently invariant under deformations of the level. However, to the best of the authors' understanding, to guarantee such invariance, sufficient control of period via action is necessary as is indicated again by the counterexamples to the Hamiltonian Seifert conjecture. Hence, this approach encounters essentially the same problem as the first one.

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